Black Holes

If enough matter from surrounding star falls back onto the Fe core so that $M_{ch}(NS)$ is exceeded, the result is a BH.

Heuristically, escape velocity for object of M and R is

$$V^2 = \frac{GM}{R} \geq c^2 \quad \text{if} \quad R < GM.$$

In GR, ~~for~~ this Schwarzchild radius $R_s = \frac{2GM}{c^2}$

or $R_s \simeq 3\,\text{km}\left(\frac{M}{M_\odot}\right)$,

is radius of event horizon, region where no light can escape from.

In GR, mass curves spacetime; so the metric around a non-spinning (Schwarzchild) BH is

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)dt^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + \frac{r^2}{c^2}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

as opposed to the Minkowski metric,

$$ds^2 = dt^2 - \frac{1}{c^2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right].$$

ds is the physical spacetime separation between pts. separated by $dt, dr, d\theta, d\phi$ at t, r, θ, ϕ.

A time interval $\Delta t'$ at distance r from the BH is observed at $r \to \infty$ to be

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{2GM}{rc^2}}}.$$

More importantly, a photon emitted at r with frequency ν_e is observed at ∞ with frequency

$$\nu_\infty = \nu_e\left(1 - \frac{2GM}{rc^2}\right)^{1/2} \simeq \nu_e\left(1 - \frac{GM}{rc^2}\right)$$

for $R \gg R_s$

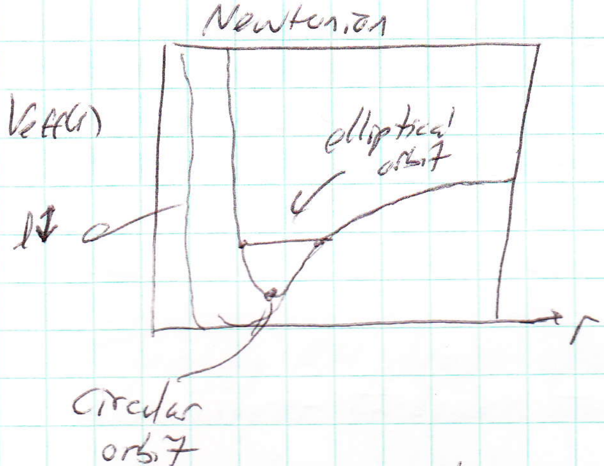
This "redshift" can be thought of simply as the energy lost by the γ as it climbs out of the grav potential well.

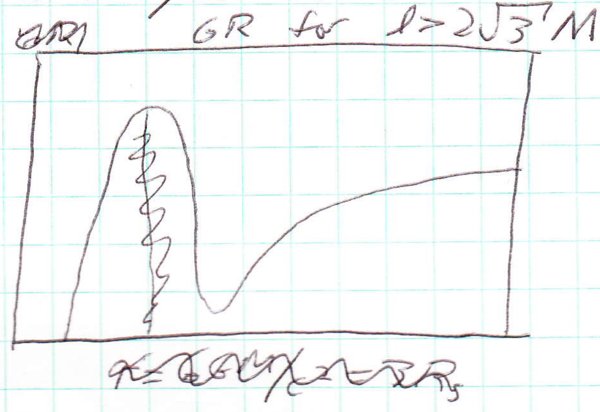
In a Newtonian $\frac{1}{r}$ potential, energy conservation yields

$$\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 - \frac{GM}{r} = V_\infty^2$$

or $$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_{eff} = V_\infty^2 \frac{1}{2} \quad \text{with} \quad V_{eff}(l) = \frac{-GM}{r} + \frac{l^2}{2r^2}$$

with $l = r^2\dot{\theta}$ = ang momentum per mass.





In Schwarzchild metric, this becomes

$$V_{eff} = \frac{-GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3c^2}$$

~~Thus V_{eff} has max at~~

$$\frac{\partial V_{eff}}{\partial r} = 0 \quad \text{when} \quad Mr^2 - l^2 r + 3Ml^2 = 0 \qquad \text{with } \left(\frac{GM}{c^2} \to M\right)$$

or at $$r = \frac{1}{2M}\left[l^2 \pm \sqrt{l^4 - 12M l^2}\right]$$

Thus, when $l \leq 2\sqrt{3}M$, there are no maxima or minima and thus no stable circular orbits.

The radius of the ~~smallest~~ innermost stable circular orbit (ISCO) is (using $l^2 = 12M^2$)

$$\boxed{r_{ISCO} = 6M}$$

The binding energy per unit mass for a particle in a circular orbit at r_{ISCO} is

$$E_{bind} = \frac{\cancel{m}-E}{m} = 1-\left(\frac{8}{9}\right)^{1/2} = 5.72\%.$$

Thus, if particle is slowly lowered to r_{ISCO} and then dropped into BH, we can extract $\sim 6\%$ of its rest-mass energy; ~~this~~ this is $\gg$ the $\sim 1\%$ obtained from nuclear burning $H \to He \to Fe$.

$\Longrightarrow$ Accretion onto BHs provides powerful energy source!

Kerr (spinning) BH:

~~BHs~~ If angular momentum conserved during collapse star $\to$ BH, the BH will be spinning very ~~fa~~ rapidly; heuristically,

$$L = I\omega = I_{in}\omega_{in} = I_{BH}\omega_{BH} \Rightarrow \omega_{BH} \propto (I_{in}/I_{BH})\,\omega_{in} \propto \left(\frac{R_*}{R_S}\right)^2 \omega_{in}.$$

Spinning BH drags spacetime around it; metric is (in Boyer-Lindquist coordinates):

$$ds^2 = \left(1-\frac{2GMr}{\rho c^2}\right)dt^2 - \frac{1}{c^2}\Big[\frac{4GMra\sin^2\theta}{\rho c}\,dt\,d\phi + \frac{\rho}{\Delta}dr^2 + \rho\,d\theta^2 + \left(r^2+a^2+\frac{2GMra^2\sin^2\theta}{\rho c^2}\right)\sin^2\theta\,d\phi^2\Big]$$

$$a = \frac{J}{Mc} \qquad \Delta = r^2 - \frac{2GMr}{c^2} + a^2 \qquad \rho = r^2 + a^2\cos^2\theta$$

$0 < a < GM/c^2$, with $a = GM/c^2$ maximally spinning BH (just as there is max ang. momentum for NS, beyond which it becomes centrifugally unbound, $\exists$ max a).

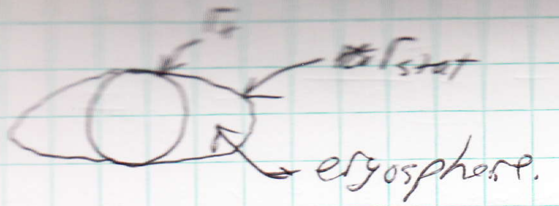
Kerr BH has horizon at $r_+ = \frac{GM}{c^2} + \left[\left(\frac{GM}{c}\right)^2 - \left(\frac{J}{Mc^2}\right)\right]^{1/2}$

For $a \to 1$ (which we expect most astrophysical BHs to have),

$$r_+ \to GM/c^2 = \frac{1}{2}R_S$$

$\Longrightarrow$ ~~Kerr BH can get deeper~~
Kerr grav potential well can be deeper

There is a static limit, $r_{stat} = \frac{GM}{c^2} + \left[\left(\frac{GM}{c^2}\right)^2 - \left(\frac{J}{Mc}\right)^2 \cos^2\theta\right]^{1/2}$



All observers within ergosphere must rotate (get dragged) with BH; they cannot be stationary.

If particle from outside ergosphere enters ergosphere and decays or scatters therein, is possible that it may come out with <u>more</u> energy/angular momentum than it started with (Penrose process); i.e., energy and angular momentum may be extracted from spinning BH; ~~to~~ up to a fraction,

$$1 - \frac{1}{\sqrt{2}}\left[1 + \left[1 + (J/J_{max})^2\right]^{1/2}\right]^{1/2} \lesssim 29\%$$

of Mc^2 may be extracted.

Likewise, energy may be extracted by threading BH with B fields (<u>Blandford-Znajek</u> mechanism).

Most important, as $a \to 1$,

$r_{ISCO} = GM/c^2$ co-rotating

$r_{ISCO} = 9GM/c^2$

42.3% of rest-mass energy ~~[illegible]~~ of accreted particle may be released

not so much ~1%.