Astrophysics of Compact Objects (171.156), Fall 2011

Problem Set 3

Due: In class, 22 September 2011

1. **Neutron-star order-of-magnitude estimates.**

   (a) **Rotation period.** Imagine our Sun collapses to a radius of 10 km. What would the final spin period be, assuming conservation of angular momentum and assuming the initial and final states are both homogeneous spheres.

   (b) **Kick velocity.** Assume that a newborn neutron star radiates 10^{53} erg in neutrinos. Show that an asymmetry in emission as small as 1% (for example, if the star radiates slightly more neutrinos in the positive-z direction than in the negative-z direction) is enough to impart a kick of several 100 km s^{-1} to the neutron star (assume a mass of 1.4 $M_\odot$). Make use of the fact that neutrinos are ultrarelativistic particles.

   (c) **Magnetic field.** Assume a star of the size and magnetic field of the Sun collapses to a radius of 10 km. Estimate the magnetic field of the resulting object, assuming that magnetic flux is conserved.

   (d) **Temperatures and accretion rates.** Consider a binary system in which matter accreting onto a compact object leads to a luminosity close to 10^{38} erg/sec, close to the Eddington luminosity for a $M_\odot$ star. Show that if the compact object is a white dwarf then it radiates in the ultraviolet, but if it is a neutron star, it radiates primarily in x-rays. Find the mass accretion rate onto a neutron star consistent with this luminosity.

   (e) **Maximum magnetic fields.** As the magnetic field in a neutron star is increased, the energy density in the magnetic field also increases. Estimate the magnetic-field strength above which the magnetic-field energy density becomes larger than the (rest-mass) energy density in the outer crust of the neutron star. Estimate the magnetic-field strength above which the magnetic-field energy density becomes larger than the (rest-mass) energy density in the core of the neutron star. Deduce from these estimates a rough upper limit to the allowed value of the magnetic field in a magnetar.

2. **Pulsar-glitch timescales.** The spinup of the crust in a pulsar glitch is communicated to the charged particles in the interior of the neutron star by the magnetic fields that thread them. Any distortion in the magnetic field due, for example, to differential rotation, generates magnetic "sound

waves," or Alfvén waves, which travel with a speed $v_A \sim (P_B/\rho)^{1/2}$, where $P_B \sim B^2/8\pi$ and ρ is the mass density. Estimate the time τ_A for the crust spinup to be communicated to the interior of the star. Find and use the magnetic-field strength typical for the Vela pulsar and compare (roughly) with the post-glitch healing time for the Vela pulsar.

3. **Pulsar glitches from magnetospheric instabilities.** Argue, using energetic considerations, that the magnetic field in the closed field-line region about a pulsar can trap charged particles up to a maximum plasma moment of inertia,
$$I_p \sim \frac{B_p^2 R^3}{6\Omega^2}. \tag{1}$$
Calculate the fractional angular-frequency change $\Delta\Omega\Omega$ if all of the plasma were released suddenly without generating a torque on the star. Evaluate your answer for the Crab and Vela pulsars, assuming reasonable values for B_p, M, and R.

4. **Hawking's area theorem.** Hawking proved that in any interaction between several black holes, the sum of the surface areas can never decrease. For a Kerr black hole of mass M and spin parameter a, the area of the horizon is
$$A = 8\pi M \left[M + (M^2 - a^2)^{1/2}\right]. \tag{2}$$
Use Hawking's area theorem to find the minimum mass M_2 of a Schwarzchild black hole that results from the collision of two Kerr black holes of equal mass M and opposite spin parameter a. Show that if $|a| \to M$, then half of the rest mass is radiated away as gravitational waves. Show that no other combinations of masses and angular momenta lead to higher possible efficiencies. Show that if $a = 0$, the maximum efficiency is 29%.

5. **Gravitomagnetism.** The frame-dragging effects of a Kerr spacetime, around a spinning black hole, can be thought in terms of gravitomagnetism, a gravitational analog magnetism. Moving masses generate in general relativity a gravitomagnetic field $\vec{B}_g$ analogously to electromagnetism. Thus, $\vec{\nabla} \times \vec{B}_g = -4\pi G\vec{J}/c$, where $\vec{J}$ is the mass current. First, estimate the gravitomagnetic field in the Solar System 1 AU from the Sun. There is then a gravito-Lorentz force $\vec{F}_m = 2m\vec{v}\times\vec{B}_g/c$ that acts on masses moving with velocity $\vec{v}$. This force acts in the Solar System on the Earth in such a way that it affects the Earth's orbital period (e.g., the period would depend on whether the Earth was co-rotating or counter-rotating with the Sun.) Estimate the magnitude of this effect.