# Quantum Mechanics (171.605), Fall 2016 

Final Exam

## Due: 4:00pm, Monday, 19 December 2016

Rules: You can spend as much time as you like, and you can refer to the book and to notes. Please do not, however, work with other students nor consult with others. Please try to avoid other resources (online, etc.), but if you do look something up, just be sure to note that in your solution.

1. If $S_{q}^{(k)}$ and $T_{q}^{(k)}$ are two irreducible-tensor operators of rank $k$, prove that

$$
\begin{equation*}
\sum_{q=-k}^{k}(-1)^{q} S_{q}^{(k)} T_{-q}^{(k)} \tag{1}
\end{equation*}
$$

is a scalar operator.
2. The magnetic-moment operator for a nucleon is $\vec{m}=e\left(g_{l} \vec{L}+g_{s} \vec{S}\right)$, where $\vec{L}$ and $\vec{S}$ are the orbital-angular-momentum and spin operators, respectively, and $g_{l}=1$ and $g_{s}=5.587$ for a proton and $g_{l}=0$ and $g_{s}=-3.826$ for a neutron. In a central field with an additional spin-orbit interaction the nucleons move in shells characterized by orbital-angular-momentum quantum number $l$ and total-angular-momentum quantum number $j=$ $l \pm \frac{1}{2}$. Calculate the magnetic moment of a single nucleon as a function of $j$ for the two kinds of nucleons, distinguishing the two cases $j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}$. Plot $j$ times the effective gyromagnetic ratio versus $j$, connecting in each case the points by straight-line segments.
3. Show that the momentum-space representation of single-particle orbital-angular-momentum states with quantum numbers $l m$ are spherical harmonics. Show that the choice of phase implied by $\langle\vec{p} \mid l m\rangle=i^{l} Y_{l}^{m}(\hat{p})$ leads to correct ahd simple time-reversal transformation properties for the angular-momentum eigenfunctions in momentum space. Compare with the time-reversal transformation properties of the orbital-angularmomentum eigenfunctions in coordinate space.
4. The $z z$ component of the quadrupole moment for a state with wave function $\psi(\vec{r})$ is

$$
\begin{equation*}
\left\langle Q_{z z}\right\rangle=\int d^{3} r|\psi(\vec{r})|^{2} r^{2} 3\left(\cos ^{2} \theta-1\right) \tag{2}
\end{equation*}
$$

Evaluate $\left\langle Q_{z z}\right\rangle /\left\langle r^{2}\right\rangle$ for the $S_{1 / 2}$ and $P_{1 / 2}$ states; for the $P_{3 / 2}$ and $D_{3 / 2}$ states (for $m_{j}= \pm 1 / 2$ and for $m_{j}= \pm 3 / 2$ ); and for the $D_{5 / 2}$ and $F_{5 / 2}$ states (for $m_{j}= \pm 1 / 2, m_{j}= \pm 3 / 2$, and $m_{j}= \pm 5 / 2$ ). Express all your answers as rational numbers.
5. Consider a plane wave of a spin- $1 / 2$ particle moving with momentum $\hbar \vec{k}$. Write down the (spinorial) wave functions $\psi_{ \pm}$for the positive- and negative-helicity states (i.e., those with $(\vec{\sigma} \cdot \hat{k}) \psi= \pm \psi)$.

