Quantum Mechanics (171.605), Fall 2016

Problem Set 4

Due: 4 October 2016

1. Do problem 2.19 in Sakurai-Napolitano.

2. This problem deals with *squeezed states* of the harmonic oscillator. In this problem we'll find a quantum state of the harmonic oscillator in which the minimum-uncertainty relation is satisfied, albeit with reduced Δq (where q is the coordinate) and increased Δp (or *vice versa*). We consider again a simple harmonic oscillator with mass m and angular frequency ω. Now define new raising and lowering operators by

$$b = \sqrt{\frac{m\omega'}{2\hbar}} \left(q + i \frac{p}{m\omega'} \right), \tag{1}$$

$$b^\dagger = \sqrt{\frac{m\omega'}{2\hbar}} \left(q - i \frac{p}{m\omega'} \right); \tag{2}$$

where ω' is some other frequency.

   (a) Verify that $[b, b^\dagger] = 1$.

   (b) Show that the operators can be written

$$b = \lambda a + \nu a^\dagger, \tag{3}$$

$$b^\dagger = \lambda a^\dagger + \nu a, \tag{4}$$

   and write λ and ν in terms of ω and ω'. Verify that that $\lambda^2 - \nu^2 = 1$ (with $\lambda > 1$).

   (c) Let $|\beta\rangle$ be an eigenstate of b: $b|\beta\rangle = \beta|\beta\rangle$. Evaluate $(\Delta q)^2$ and $(\Delta p)^2$ for this state. Show that although either can be made arbitrarily small, we still always maintain $\Delta q \Delta p = \hbar/2$.

   (d) Show that the operators b and a are related by a unitary transformation, $b = UaU^\dagger$, where

$$U = e^{\zeta[a^2 - (a^\dagger)^2]/2}, \tag{5}$$

   and $e^\zeta = \lambda + \nu$.

   (e) Suppose a coherent state $|\alpha\rangle$ (an eigenstate of a) is transformed into a squeezed state by the unitary operator,

$$U = \exp\left\{ \frac{\zeta}{2} \left[a^2 - (a^\dagger)^2 \right] \right\}. \tag{6}$$

Calculate the value of ζ that will reduce the width of the Hermitian observable $(a + a^\dagger)/\sqrt{2}$ to 1 percent of its original coherent-state value. What happens to the width of the conjugate observable $(a - a^\dagger)/\sqrt{2}i$ in this transformation?

(f) If you're interested, browse online to learn how squeezed states are useful in quantum optics.

3. Do also problems 2.20, 2.22, and 2.24.