Particle Astrophysics 171.697
Problem Set 5
Due: First class of week 6

Preview: Though the subject of this week is homogeneous inflationary dynamics, similar equations also describe the evolution of a quintessence field, a candidate for the dark energy today. This problem set thus also deals a bit with the latter subject.

1. **A $w = 1$ equation of state from a rolling scalar field.** Consider a massless scalar field; i.e., a scalar field $\phi(\vec{x}, t)$ whose potential-energy density is $V(\phi) = 0$. Now suppose that this scalar field is initially rolling, so $\dot{\phi} \neq 0$, and that the kinetic-energy density associated with this rolling dominates the energy density of the Universe. Show from the stress-energy tensor $p = \rho$ for this type of matter. Show that this implies that $\rho \propto a^{-6}$, where a is the scale factor, in two ways: (1) by recalling how the energy density of matter with an equation of state $p = w\rho$ scales with a; and (2) by solving the equation of motion for ϕ in an expanding Universe. (This should be a very simple problem.)

2. (From LL 3.7) **Phenomenology of $\lambda\phi^4$ inflation.** Consider $V(\phi) = \lambda\phi^4$, where λ is the self-coupling. Assume that the field rolls toward $\phi = 0$ from the positive side. Calculate the value of ϕ where each of the slow-roll conditions (i.e., $\epsilon \ll 1$ and $\eta \ll 1$) first break down. Do they break down at the same place? Assuming that inflation ends when $\epsilon = 1$, calculate the number of e-foldings of inflation that occur for an initial value ϕ_i. Demonstrate that the slow-roll solutions with $\phi = \phi_i$ and $a = a_i$ at $t = t_i$ are

$$\phi = \phi_i \left[-\sqrt{\frac{32\lambda M_{\mathrm{Pl}}^2}{6}} (t - t_i) \right],$$

$$a = a_i \exp\left(\frac{\phi_i^2}{8M_{\mathrm{Pl}}^2} \left\{ 1 - \exp\left[-\sqrt{\frac{64\lambda M_{\mathrm{Pl}}^2}{3}} (t - t_i) \right] \right\} \right).$$

Use the solution for ϕ to calculate the time that inflation ends. Demonstrate that the number of e-foldings calculated using the solution for a is the same as that which you calculated above. Expand the solution for a at small $t - t_i$ to demonstrate that the inflation is approximately exponential in the initial stage. Calculate the time constant κ [from $a \sim \exp(\kappa t)$] and demonstrate that it equals the (slow-roll) Hubble parameter during inflation.

3. **Tracker field.** Consider a scalar field that rolls down a potential-energy density $V(\phi) = V_0 e^{-\phi/\phi_0}$. Now suppose that the energy density of the Universe is dominated by ordinary non-relativistic matter (so $a \propto t^{2/3}$), and that the energy density of the rolling scalar field is negligible compared with the non-relativistic matter. Show that there is a solution to the scalar-field equation of motion such that the energy density $\rho_\phi = (1/2)\dot{\phi}^2 + V(\phi)$ of

the scalar field scales as $\rho_\phi \propto a^{-3}$, the same as the ordinary matter. Does the same thing happen if the energy density of the Universe is dominated by relativistic matter? This is the basis for the "tracker-field" solutions that have been discussed in the literature recently.

4. **The monopole problem.** Calculate the relic density of magnetic monopoles, assuming that there is one GUT-mass ($\sim 10^{15}$ GeV) monopole produced per Hubble volume at the GUT phase transition ($T \sim 10^{15}$ GeV). You should get an unreasonably large number. There is a bound $\Omega_{\rm monopole} \lesssim 10^{-6}$ (the Parker bound) to the relic density of magnetic monopoles in the Universe today. Calculate the number of e-folds of inflation after the GUT transition required to solve the monopole problem.

5. **Anharmonic scalar-field oscillations.** In class we argued that if we have a real scalar field ϕ with a quadratic potential $V(\phi) = (1/2)m^2H^2$, and if $m \gtrsim H$ (implying that the oscillation frequency is large than the expansion rate), then coherent oscillations of the scalar field imply that the pressure $p = 0$ when averaged over an oscillation cycle and thus that the energy density $\rho \propto a^{-3}$. Now consider oscillations in a potential $V(\phi) = c|\phi|^n$, where c is a constant. Show that coherent oscillations in such a potential give rise to an energy density that decays as $\rho \propto a^{-\alpha}$, and determine α. Of course, you should recover $\alpha = 3$ for $n = 2$. What value of n is required to produce $\alpha = 4$ (i.e., radiation)? Can you think of a physical argument that justifies your result? Likewise, is there a value of n that produces $\alpha = 0$? Can you explain this result in physical terms?