Chapter 7

CMB polarization

7.1 The polarization tensor

The intensity of the radiation is proportional to the electric field squared, $I \propto \langle E^2 \rangle$. We can generalize this scalar quantity to a symmetric tensor, $I_{ab} \propto \langle E_a E_b \rangle$. Because the electromagnetic fields are orthogonal to the direction of propagation of a a wave, $\mathbf{I}(\hat{n}) \cdot \hat{n} = \mathbf{0}$. The scalar intensity is just the trace of this tensor: $I \equiv I_{aa}$. The remainder is the *polarization tensor*:

$$I_{ab} = \frac{I}{2}(\delta_{ab} - n_a n_b) + P_{ab}. \tag{7.1}$$

$P_{ab}(\hat{n})$ a 3 by 3 symmetric and trace-free tensor orthogonal to the direction of propagation, $P_{ab}n^b = 0$. This means that P_{ab} has only two independent components. Since it is symmetric, we can diagonalize it. We already know that $\hat{n}$ is one of the eigenvectors (with eigenvalue 0). Let us denote by $\hat{u}$ and $\hat{v}$ the remaining 2 eigenvectors, orthogonal to $\hat{n}$. Given that P_{ab} is traceless, it must take the following form in the basis $(\hat{u}, \hat{v}, \hat{n})$:

$$P_{ab}(\hat{n}) = \lambda(u_a u_a - v_a v_a) = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.2}$$

This represents a radiation field polarized along $\pm\hat{u}$, with total polarization λ. If we denote by $\hat{x}$ and $\hat{y}$ a fixed basis orthogonal to $\hat{n}$, and $\hat{u} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$ and $\hat{v} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$, then we have, in the basis $(\hat{x}, \hat{y}, \hat{n})$,

$$P_{ab}(\hat{n}) = \lambda\left[\cos(2\varphi)(\hat{x}_a\hat{x}_b - \hat{y}_a\hat{y}_b) + \sin(2\varphi)(\hat{x}_a\hat{y}_b + \hat{y}_a\hat{x}_b)\right] = \lambda \begin{pmatrix} \cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} Q & U & 0 \\ U & -Q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.3}$$

where we used the *Stokes parameters* $Q \equiv \lambda\cos(2\varphi)$ and $U \equiv \lambda\sin(2\varphi)$. Thus we see that the two independent numbers describing P_{ab} are the angle φ and the magnitude of polarization λ, or equivalently Q and U. We note that these parameters only describe linear polarization, which is the only relevant kind of polarization here. In terms of the electric field, we have

$$I \propto \langle E_x^2 \rangle + \langle E_y^2 \rangle, \tag{7.4}$$

$$Q \propto \langle E_x^2 \rangle - \langle E_y^2 \rangle, \tag{7.5}$$

$$U \propto 2\langle E_x E_y \rangle. \tag{7.6}$$

7.2 Thomson scattering

7.2.1 General expression

Given an incoming intensity tensor $\mathbf{I}(n')$, the simplest guess for the intensity tensor after scattering in the direction n is the projection of $\mathbf{I}(n')$ perpendicular to n:

$$I_{ab}(n) \propto (\mathbf{I}'_{\perp n})_{ab} \equiv I'_{ab} - I'_{ac}n_c n_b - I'_{bc}n_c n_a + I'_{cd}n_c n_c n_a n_b \tag{7.7}$$

You can check for yourself that this tensor is orthogonal to n; it is also obviously symmetric. It also implies that, for an incoming radiation polarized along $\pm\boldsymbol{\epsilon}'$, the scattered intensity along $\pm\boldsymbol{\epsilon}$ is proportional to $(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2$.

The rate of change of the photon intensity due to Thomson scattering therefore takes the form

$$\frac{d\mathbf{I}}{dt}\Big|_{\rm Th} = n_e\sigma_{\rm T}\left[C\int dn'\mathbf{I}'_{\perp n} - \mathbf{I}(n)\right], \tag{7.8}$$

where we now determine the constant C. Taking the trace, we have

$$\mathrm{Tr}(\mathbf{I}'_{\perp n}) = I' - I'_{ab}n_a n_b, \tag{7.9}$$

therefore

$$\frac{dI}{dt}\Big|_{\rm Th} = n_e\sigma_{\rm T}\left[C\int dn'(I' - I'_{ab}n_a n_b) - I(n)\right], \tag{7.10}$$

Thomson scattering does not change the photon energy (in the limit we are considering), neither the total number of photons, and as a consequence the angle-integrated I must be conserved. You can check that this implies $C = \frac{3}{8\pi}$.

7.2.2 Total intensity

Using Eq. (7.1) we obtain, defining $d\tau \equiv n_e\sigma_{\rm T}dt$,

$$\frac{dI}{d\tau}\Big|_{\rm Th} = \frac{3}{16\pi}\int dn'\left[1 + (n\cdot n')^2\right]I' - I(n) - \frac{3}{2}n_a n_b\int\frac{dn'}{4\pi}P'_{ab}. \tag{7.11}$$

We have already encountered the first two terms when studying temperature-only perturbation; we can see that in addition, the angle-averaged polarization tensor also sources total intensity.

In practice we work with the temperature perturbation Θ such that $I \propto \overline{T}^3(1+\Theta)$, and with the polarization in units of $\overline{T}^3$ (we keep the same notation for that quantity, P_{ab}):

$$\frac{d\Theta}{d\tau}\Big|_{\rm Th} = \frac{3}{16\pi}\int dn'\left[1 + (n\cdot n')^2\right]\Theta' - \Theta(n) - \frac{3}{2}n_a n_b\int\frac{dn'}{4\pi}P'_{ab}. \tag{7.12}$$

7.2.3 Polarization

The rate of change of the polarization tensor is the transverse trace-free part of the rate of change of the intensity tensor:

$$\frac{d\mathbf{P}}{dt}\Big|_{\rm Th} = \left(\frac{d\mathbf{I}}{dt}\Big|_{\rm Th}\right)_{\rm tt}, \tag{7.13}$$

where

$$(X_{\rm tt})_{ab} \equiv X_{ab} - \frac{X_{cc}}{2}(\delta_{ab} - n_a n_b). \tag{7.14}$$

We need to compute the transverse trace-free part of $\mathbf{I}'_{\perp n}$. You can easily check that the part proportional to the identity matrix $\mathbf{1}$ (with components δ_{ab}) does not contribute, so we have

$$\mathbf{I}'_{\perp n,\rm tt} \;=\; -\frac{I'}{2}(n'\otimes n')_{\perp n,\rm tt} + \mathbf{P}'_{\perp n,\rm tt} = -\frac{I'}{2}\left(n'\otimes n' - \frac{1}{3}\mathbf{1}\right)_{\perp n,\rm tt} + \mathbf{P}'_{\perp n,\rm tt}, \tag{7.15}$$

where $(n\otimes n)_{ab} \equiv n_a n_a$ and we have added a (for now seemingly arbitrary) part times the identity matrix, whose $(\perp n, \rm tt)$ part vanishes. So we get

$$\frac{d\mathbf{P}}{d\tau}\Big|_{\rm Th} = -\frac{3}{16\pi}\int dn'\frac{I'}{2}\left(n'\otimes n' - \frac{1}{3}\mathbf{1}\right)_{\perp n,\rm tt} + \frac{3}{8\pi}\int dn'\mathbf{P}'_{\perp n,\rm tt} - \mathbf{P}. \tag{7.16}$$

We can intervert the order of integrating and projecting and rewrite this as

$$\frac{d\mathbf{P}}{d\tau}\Big|_{\rm Th} = -\frac{3}{32\pi}\left[\int dn' I(n')\left(n'\otimes n' - \frac{1}{3}\mathbf{1}\right)\right]_{\perp n,\rm tt} + \frac{3}{2}\left[\int\frac{dn'}{4\pi}\mathbf{P}'\right]_{\perp n,\rm tt} - \mathbf{P}. \tag{7.17}$$

The intensity $I(n')$ can be decomposed in its multipole moments. This is usually done using a spherical harmonic decomposition in a specific basis, but we can also do it in a coordinate-independent fashion:

$$I(n') = I^{(0)} + I^{(1)}_a n'_a + I^{(2)}_{ab}\left(n'_a n'_b - \frac{1}{3}\delta_{ab}\right) + ..., \tag{7.18}$$

where $I^{(0)}$, $I^{(1)}_a$, $I^{(2)}_{ab}$ are the monopole, dipole, and quadrupole moment, respectively, and given by [check normalization]

$$I^{(0)} \equiv \int \frac{dn'}{4\pi} I(n'), \tag{7.19}$$

$$I^{(1)}_a \equiv 3\int \frac{dn'}{4\pi} n'_a I(n'), \tag{7.20}$$

$$I^{(2)}_{ab} \equiv \frac{15}{2}\int \frac{dn'}{4\pi}\left(n'_a n'_b - \frac{1}{3}\delta_{ab}\right) I(n') = \frac{15}{8\pi}\int dn' I(n')\left(n' \otimes n' - \frac{1}{3}\mathbf{1}\right)_{ab}. \tag{7.21}$$

The piece $\frac{1}{3}\delta_{ab}$ in the quadrupole moment is there so that the quadrupole piece averages to zero (it is orthogonal to the constant monopole). Rewriting things in terms of the temperature perturbation Θ (and now rescaling P_{ab}), we therefore arrive at

$$\left.\frac{d\mathbf{P}}{d\tau}\right|_{\rm Th} = -\frac{1}{20}\mathbf{\Theta}^{(2)}_{\perp n,\rm tt} + \frac{3}{2}\left[\int \frac{dn'}{4\pi}\mathbf{P}'\right]_{\perp n,\rm tt} - \mathbf{P}, \tag{7.22}$$

$$\Theta(n') = \Theta^{(0)} + \Theta^{(1)}_a n'_a + \Theta^{(2)}_{ab}\left(n'_a n'_b - \frac{1}{3}\delta_{ab}\right) + ..., \tag{7.23}$$

We therefore see that the *quadrupole* of the intensity sources the polarization tensor (note that our definition of the quadrupole may not be the standard one, but the combination of the two equations is all that matters). This can be understood intuitively graphically, including the minus sign.

Just like for the temperature perturbation, we can write a line-of-sight solution for the polarization tensor. In both cases, the visibility function (which gives the probability of last scattering) has two bumps: a large one at recombination and a smaller one at reionization. The total probability of last scattering at reionization is $1 - e^{-\tau_{\rm reion}} \approx \tau_{\rm reion} \approx 0.06$ (according to the latest *Planck* analysis). The source function for the polarization contains the local intensity quadrupole.

7.3 E and B modes

A vector field $\vec{V}(\vec{x})$ can be decomposed into a gradient mode G and a curl mode $\vec{C}$:

$$\vec{V} = \vec{\nabla} G - \vec{\nabla}\times\vec{C}, \qquad \vec{\nabla}\cdot\vec{C} = 0 \tag{7.24}$$

The components G and $\vec{C}$ are uniquely defined by

$$\nabla^2 G \equiv \vec{\nabla}\cdot\vec{V}, \qquad \nabla^2\vec{C} \equiv \vec{\nabla}\times\vec{V}. \tag{7.25}$$

The same can be done for a polarization map $P_{ab}(\hat{n})$ on the sky. We first define the vector $V_a \equiv \nabla^b_\perp P_{ab}$, where $\vec{\nabla}_\perp$ is the gradient perpendicular to the line of sight (to formalize this better, one could introduce an arbitrary angular diameter distance that cancels out in the definition of E and B):

$$\nabla^a_\perp = (\delta^{ab} - n^a n^b)\nabla_b. \tag{7.26}$$

The vector $\vec{V}$ is itself perpendicular to the line of sight:

$$n_a(\nabla^b_\perp P_{ab}) = \nabla^b_\perp(n_a P_{ab}) - P_{ab}\nabla^b_\perp n_a = \nabla^b_\perp(\ 0\ ) - P_{ab}(\delta^{ab} - n^a n^b) = 0. \tag{7.27}$$

We then define the gradient (E) and curl (B) modes as follows:

$$\nabla^2_\perp E \equiv \nabla_\perp\cdot\vec{V}, \qquad \nabla^2_\perp B \equiv \hat{n}\cdot(\vec{\nabla}_\perp\times\vec{V}). \tag{7.28}$$

No B-modes for scalar initial conditions

For *scalar* initial conditions, with initial perturbations $\zeta(\boldsymbol{k})$, any perturbation is proportional to $\zeta(\boldsymbol{k})$. Since the coefficients of the Boltzmann equation only depends on k and $\mu \equiv \hat{k}\cdot\hat{n}$, for a given $\boldsymbol{k}$, the only geometric form the polarization tensor may take is

$$\mathbf{P}(\boldsymbol{k}, n) \propto (\hat{k}\otimes\hat{k})_{\perp n,\mathrm{tt}}\ \zeta(\boldsymbol{k}), \tag{7.29}$$

where the coefficient only depends on k and μ. We note that

$$\left[(\hat{k}\otimes\hat{k})_{\perp n,\mathrm{tt}}\right]_{ab} = (\hat{k}_a - \mu n_a)(\hat{k}_b - \mu n_b) - \frac{1}{2}(1-\mu^2)(\delta_{ab} - n_a n_b) = \frac{1}{2}(1-\mu^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.30}$$

in the basis where the first vector is along the component of $\hat{k}$ perpendicular to n, the second vector is orthogonal to $\hat{k}$ and n, and the third vector is n. So we may write

$$\mathbf{P}(\boldsymbol{k}, n) = \frac{2\Delta_P(k,\mu)}{1-\mu^2}(\hat{k}\otimes\hat{k})_{\perp n,\mathrm{tt}}\ \zeta(\boldsymbol{k}). \tag{7.31}$$

With this convention, the polarization amplitude is $\Delta_P(k,\mu)\zeta(\boldsymbol{k})$. In words, with scalar initial conditions, the polarization must be along the the component of $\boldsymbol{k}$ orthogonal to $\hat{n}$, and its amplitude only depends on the angle between $\hat{k}$ and $\hat{n}$.

Similarly, the vector V_a is linearly related to $\zeta(\boldsymbol{k})$ and is orthogonal to $\hat{n}$. It therefore takes the form

$$V_a = V(k,\mu)(\hat{k}_a - \mu n_a)\zeta(\boldsymbol{k}). \tag{7.32}$$

Therefore

$$\nabla_\perp^b V_a = \left[\partial_\mu V(\hat{k}_a - \mu n_a) - V n_a\right](\hat{k}_b - \mu n_b)\zeta(\boldsymbol{k}) - V\mu(\delta_{ab} - n_a n_b)\zeta(\boldsymbol{k}). \tag{7.33}$$

This implies

$$\nabla_\perp^2 B = n_c \epsilon_{cba}\nabla_\perp^b V_a = -V n_c \epsilon_{cba} n_a \hat{k}_b \zeta(\boldsymbol{k}) = 0. \tag{7.34}$$

In words, the vector $\vec{V}$ is everywhere parallel to the component of $\vec{k}$ perpendicular to the line of sight, and as a consequence, its curl vanishes. This holds for a single $\boldsymbol{k}$ mode, and by linearity, for arbitrary scalar initial conditions. A B-mode contribution to the polarization pattern is therefore a tell-tale of primordial *tensor* (or vector) modes.

7.4 Scalar perturbations

7.4.1 Polarization Boltzmann equation

Let us consider scalar initial conditions, and suppose they are entirely determined by a single scalar field $\zeta(\boldsymbol{k})$ – this is the case for *adiabatic* initial conditions. We define $P(k,\mu)$ as in Eq. (7.31). We then have

$$\hat{k}_a\hat{k}_b P_{ab}(\boldsymbol{k}, n) = (1-\mu^2)\Delta_P(k,\mu)\zeta(\boldsymbol{k}). \tag{7.35}$$

We also have

$$\int \frac{dn'}{4\pi} P_{ab}(\boldsymbol{k}, n') = \frac{1}{2}\int_{-1}^{1} d\mu'(1-\mu'^2)\Delta_P(k,\mu')\hat{k}_a\hat{k}_b\zeta(\boldsymbol{k}). \tag{7.36}$$

A similar geometric argument implies that $\Theta_{ab}^{(2)} = \Theta_2\ \hat{k}_a\hat{k}_b\zeta(\boldsymbol{k})$. Therefore,

$$\left(\Theta^{(2)}_{\perp n,\mathrm{tt}}\right)_{ab}\hat{k}_a\hat{k}_b = \frac{1}{2}(1-\mu^2)^2\Theta_2\zeta(\boldsymbol{k}). \tag{7.37}$$

The Boltzmann equation for Δ_P is then obtained by dotting that of $\mathbf{P}$ with $\hat{k}_a\hat{k}_b$:

$$\dot{\Delta}_P + i(\boldsymbol{k}\cdot\hat{n})\Delta_P = \dot{\tau}\left[-\frac{1}{40}(1-\mu^2)\Theta_2 + \frac{3}{4}(1-\mu^2)\frac{1}{2}\int_{-1}^{1} d\mu'(1-\mu'^2)\Delta_P(k,\mu') - \Delta_P(\boldsymbol{k},\mu)\right], \tag{7.38}$$

where we have included the transport terms in the left-hand side. The way to solve this equation numerically is to decompose $\Delta_P(k,\mu)$ in the basis of Legendre polynomials.

7.4.2 Temperature quadrupole in the tight-coupling limit

We recall that the Boltzmann equation for the temperature perturbation Θ, in the Newtonian gauge:

$$\dot{\Theta}(\hat{n}) + i(\boldsymbol{k}\cdot\hat{n})(\Theta(\hat{n})+\psi) - \dot{\phi} = \dot{\tau}\left[\tilde{\Theta}(\hat{n}) + \boldsymbol{v}_b\cdot\hat{n} - \Theta(\hat{n}) - \frac{3}{2}n_a n_b \int \frac{dn'}{4\pi}P'_{ab}\right], \tag{7.39}$$

$$\tilde{\Theta}(\hat{n}) \equiv \int dn' \frac{3}{16\pi}[1+(\hat{n}\cdot\hat{n}')^2]\Theta(\hat{n}'), \qquad \dot{\tau}\equiv a n_e \sigma_{\rm T}. \tag{7.40}$$

The term proportional to the baryon velocity accounts for the anisotropic radiation field seen by moving baryons, even if the CMB is initially isotropic.

In the tight coupling limit $\dot{\tau}\gg k$, the term in brackets in the right-hand side must almost vanish:

$$\Theta(\hat{n}) \approx \tilde{\Theta}(\hat{n}) + \vec{v}_b\cdot\hat{n}, \tag{7.41}$$

where we neglected the polarization piece, which, as we shall see, is small. In particular, taking the quadrupole component of this equation implies that the quadrupole of Θ must nearly vanish. We therefore write

$$\Theta(\hat{n}) = \Theta_0 + \vec{v}_b\cdot\hat{n} + \epsilon(\hat{n}), \tag{7.42}$$

where ϵ is a small correction. Substituting into the Boltzmann equation we get

$$\epsilon \approx -\frac{1}{\dot{\tau}}\left[\dot{\Theta}_0 + \dot{\vec{v}}_b\cdot\hat{n} - \dot{\phi}\right] - i\frac{\boldsymbol{k}\cdot\hat{n}}{\dot{\tau}}\left[\Theta_0 + \vec{v}_b\cdot\hat{n}\right]. \tag{7.43}$$

In the tight-coupling regime, $\vec{v}_b\approx\vec{v}_\gamma$. For scalar initial conditions,

$$\vec{v}_\gamma = \frac{\hat{k}}{k}\vec{k}\cdot\vec{v}_\gamma = 3\frac{\hat{k}}{k}i\dot{\Theta}_0, \tag{7.44}$$

where we have used the continuity equation. Therefore,

$$i(\boldsymbol{k}\cdot\hat{n})\vec{v}_b\cdot\hat{n} \approx -3(\hat{k}\cdot\hat{n})^2\dot{\Theta}_0. \tag{7.45}$$

So we find that the quadrupolar piece of Θ is

$$\Theta_{\rm quad} \approx \frac{1}{\dot{\tau}}\left[1-3(\hat{k}\cdot\hat{n})^2\right]\dot{\Theta}_0 = -\frac{3}{\dot{\tau}}\hat{k}_a\hat{k}_b\left(\hat{n}_a\hat{n}_b - \frac{1}{3}\delta_{ab}\right)\dot{\Theta}_0. \tag{7.46}$$

From this we read off what we defined as the quadrupole:

$$\Theta^{(2)}_{ab} = -\frac{3}{\dot{\tau}}\hat{k}_a\hat{k}_b\dot{\Theta}_0, \tag{7.47}$$

and as a consequence the source term in the polariation equation is proportional to $\dot{\Theta}_0/\dot{\tau}$.

Now, in the tight-coupling limit, photons undergo acoustic oscillations, so $\Theta_0 \propto \cos(kc_s\eta)$. Hence, the quadrupole is proportional to

$$\Theta_2 \propto \frac{k}{\dot{\tau}}\sin(kc_s\eta). \tag{7.48}$$

This implies the following qualitative features for the polarization power spectrum from last-scattering around recombination:

(*i*) the acoustic peaks are out-of-phase with those of the temperature power spectrum: peaks in the temperature correspond to troughs in the polarization.

(*ii*) the polarization power spectrum (from recombination) is suppressed by $\sim(k/\dot{\tau})^2$ relative to the temperature. This implies a rapid drop at large scales.

(*iii*) as a consequence, the contribution of re-scattering at reionization has a larger relative amplitude, and shows as a "bump" at $\ell\leq 10$. The large-scale polarization power spectrum is what most strongly constrains the optical depth to reionization.

Note that there is also a TE cross correlation.

7.5 Tensor perturbations

Let us now consider tensor metric perturbations. Consider the metric

$$ds^2 = a^2 \left[-d\eta^2 + (\delta_{ij} + h_{ij} dx^i dx^j)\right], \tag{7.49}$$

where h_{ij} is a symmetric, traceless, and transverse tensor ,i.e. for each Fourier mode, $k^i h_{ij}(\boldsymbol{k}) = 0$.

7.5.1 Einstein's equation

Einstein's equations take the form

$$\ddot{h}_{ij} + 2aH\dot{h}_{ij} + k^2 h_{ij} = 8\pi G a^2 \Sigma_{ij}, \tag{7.50}$$

where Σ_{ij} is the *anisotropic stress*, proportional to the photon (and neutrino) intensity quadrupole.

In the tight-coupling regime (and in the absence of neutrinos), the anisotropic stress is small, and the tensor satisfies a (damped) wave equation

$$\ddot{h}_{ij} + 2aH\dot{h}_{ij} + k^2 h_{ij} \approx 0. \tag{7.51}$$

During radiation domination, $a \propto \eta$, so $aH = 1/\eta$ and this equation becomes

$$\ddot{h}_{ij} + \frac{2}{\eta}\dot{h}_{ij} + k^2 h_{ij} \approx 0. \tag{7.52}$$

The solution that is well-behaved at $\eta \to 0$ is

$$h_{ij} = \zeta_{ij} \frac{\sin(k\eta)}{k\eta}. \tag{7.53}$$

During matter domination, $a \propto \eta^2$, so $aH = 2/\eta$ and we have

$$\ddot{h}_{ij} + \frac{4}{\eta}\dot{h}_{ij} + k^2 h_{ij} \approx 0. \tag{7.54}$$

This has the solution

$$h_{ij} = \frac{C_1}{(k\eta)^2}\left[\cos(k\eta) - \frac{\sin(k\eta)}{k\eta}\right] + \frac{C_2}{(k\eta)^2}\left[\sin(k\eta) + \frac{\cos(k\eta)}{k\eta}\right]. \tag{7.55}$$

For modes that enter the horizon deep inside the radiation era $k \gg k_{\rm eq}$, h_{ij} is initially constant then decays as $\sin(k\eta)/(k\eta)$, and then, after matter-radiation equality, decays as $e^{\pm ik\eta}/(k\eta)^2$. For modes that enter the horizon deep inside the matter-dominated era ($k \ll k_{\rm eq}$), we must set $C_2 = 0$.

In either case, tensor modes are initially constant, and decay as $1/a$ after horizon entry, while oscillating.

7.5.2 Boltzmann equation

The change of photon momentum due to gravitational waves is

$$\dot{q} = \frac{1}{2} q n_i n_j \dot{h}_{ij}, \tag{7.56}$$

where n_i is the direction of propagation. The Boltzmann equation for photon temperature perturbations therefore becomes (if there are only tensor modes, implying in particular that $\boldsymbol{v}_b = 0$)

$$\dot{\Theta}(\hat{n}) + i(\boldsymbol{k}\cdot\hat{n})\Theta(\hat{n}) - \frac{1}{2}n_i n_j \dot{h}_{ij} = \dot{\tau}\left[\tilde{\Theta}(\hat{n}) - \Theta(\hat{n}) - \frac{3}{2}n_a n_b \int \frac{dn'}{4\pi} P'_{ab}\right]. \tag{7.57}$$

In the tight-coupling limit, we can extract the temperature quadrupole just like we did in the case of the scalars, and find

$$\Theta^{(2)}_{ij} \approx \frac{1}{2\dot{\tau}}\dot{h}_{ij}. \tag{7.58}$$

For modes which are still super-horizon at recombination, $\dot{h}_{ij} \propto k\zeta_{ij}$, so power is suppressed on super-horizon scales at recombination (but here also, there is a reionization bump). For modes that are sub-horizon at recombination, $\dot{h}_{ij} \propto 1/k$, so here also there is a suppression of power. To conclude, the polarization power spectrum induced by gravitational waves peakes around the horizon scale at recombination, and decays on either side, with oscillations on small scales.