Week 8: Evolution of Perturbations

March 27, 2017

1 Kinetic Theory

We will follow Ch. 6 in Weinberg's book quite closely. As seen therein, the exact calculations for the evolution of radiation perturbations require that we take into account the fact that Thomson scattering of photons depends on the photon polarization. To simplify and get to the basic physics results, we will assume that photons are unpolarized and that Thomson scattering is insensitive to polarization. We will also consider only scalar perturbations—the treatment of tensor perturbations is then analogous.

We consider an individual Fourier mode with comoving wavenumber $\vec{q}$. The value of any quantity $X(\vec{x}, t)$ is then

$$\int \frac{d^3q}{(2\pi)^3} X(t)\alpha(\vec{q})e^{i\vec{q}\cdot\vec{x}}, \tag{1}$$

where $X(t)$ encodes the time dependence, and we are assuming that the time dependence is dominated by a growing mode. Here, $\alpha(\vec{q})$ is an amplitude for this mode.

We work in synchronous gauge in which case the metric perturbation is

$$\delta g_{ij}(\vec{x}, t) = \int \frac{d^3q}{(2\pi)^3}\, \alpha(\vec{q})\left[A_q(t)\delta_{ij} - q_i q_j B_q(t)\right] e^{i\vec{q}\cdot\vec{x}}, \tag{2}$$

where $A(t)$ and $B(t)$ encode the time dependence of the two different synchronous-gauge scalar potentials.

We assume that the dark matter is cold—i.e., that the particles have no thermal velocities and that they interact with each other and everything else only gravitationally. The density-perturbation amplitude $\delta\rho_{Dq}$ for therefore satisfies the continuity equation,

$$\delta\dot{\rho}_{Dq} + 3H\delta\rho_{Dq} = -\rho_D\psi_q, \tag{3}$$

where $\psi_q \equiv (3\dot{A}_q - q^2\dot{B}_q)/2$ is the gravitational-acceleration term in synchronous gauge.

The analogous equation for baryons is (suppressing the q subcsripts for notational economy),

$$\delta\dot{\rho}_B + 3H\delta\rho_B - (q^2/a^2)\bar{\rho}_B\delta u_B = -\bar{\rho}_B\psi_q; \tag{4}$$

The new term here arises as a consequence of the possibility that baryons may be coupled to the photons (as we will shortly see), and therefore they do not necessarily fall freely in the gravitational field, as do the dark-matter particles. The combined photon-baryon plasma satisfies the momentum-conservation equation which can be written,

$$\delta p_\gamma - q^2 \pi_\gamma^s + (\partial_0 + 3H)\left[\bar{\rho}_B \delta u_b + \frac{4}{3}\bar{\rho}_\gamma \delta u_\gamma\right] = 0. \tag{5}$$

The description of the photons is far more complicated. Far before decoupling the photons are tightly coupled through Thomson scattering to the baryons and so $\delta u_\gamma = \delta u_B$, $\delta p_\gamma = \delta\rho_\gamma/3$, and $\pi_\gamma^S = 0$. But later, we have to take into account the fact that photons can Thomson scatter from electrons. We also note that photons move at the speed of light and can therefore carry momentum from one point to another—i.e., they can develop significant shear stresses, particularly near the time of CMB decoupling when their mean-free paths become large. As a result, we must derive a *Boltzmann equation* which describes the time evolution of the completion photon phase-space distribution $n(\vec{x},\vec{p},t)$. This is defined so that $n(\vec{x},\vec{p},t)d^3x d^3p$ is the number of photons, at time t in the physical volume d^3x centered at x and the momentum-space volume d^3p centered on p. The Boltzmann equation is a phase-space continuity equation; it says that the number of photons in any given differential phase-space volume can increase when photons enter it and decrease when photons exit. Photons are constantly moving into and out of any given physical volume from and to adjacent volumes. They are re-distributed in momentum space by Thomson scattering. They are also re-distributed in momentum space by perturbations to the spacetime metric. Schematically, the Boltzmann equation is

$$\frac{D}{Dt} n(\vec{x},\vec{p},t) = [\text{Collisions}]\,. \tag{6}$$

Here (D/Dt) is a (Lagrangian) time derivative taken along the photon trajectory, and the right hand side describes the removal of photons from scattering to other parts of phase space and the addition of photons due to scattering from other parts of phase space. In the absence of collisions, Liouville's theorem is satisfied; i.e., the phase-space density is conserved along a photon trajectory. In perturbed Universe, the time derivative is taken along a geodesic, which satisfies the geodesic equation, and so the metric perturbation variables show up in the Boltzmann equation through D/Dt.

Weinberg provides a complete derivation of the Boltzmann equation. It is important to work through on your own, but it is too long to go through in class, and I'm not sure there's much value added to going through it in class. Here, I'll summarize the main points.

We write the phase-space distribution as

$$n(\vec{x},\vec{p},t) = \bar{n}_\gamma\left(a(t)p^0(\vec{x},\vec{p},t)\right) + \delta n(\vec{v},\vec{p},t), \tag{7}$$

in terms of homogeneous component and a perturbation. Here,

$$p^0 \equiv \sqrt{g^{ij}(\vec{x},t)p_i p_j}, \tag{8}$$

and

$$\bar{n}_\gamma = \left[\exp\left(p/a(t)\bar{T}(t)\right) - 1\right]^{-1}. \tag{9}$$

The photon energy-momentum tensor (which we need as input to the Einstein equations that determine the scalar potentials) is

$$T^{\mu}_{\gamma\,\nu} = \frac{1}{\sqrt{Detg}} \int \frac{d^3p}{(2\pi)^3} n \frac{p^\mu p_\nu}{p^0}. \tag{10}$$

The perturbations to this tensor can be written in terms of a dimensionless intensity J by,

$$J(\vec{x}, \hat{p}, t) \equiv \frac{1}{a(t)\bar{\rho}_\gamma(t)} \int_0^\infty \delta n_\gamma(\vec{x}, p\hat{p}, t) 4\pi p^3 \, dp, \tag{11}$$

as follows:

$$\delta T^{i}_{\gamma\,j}(\vec{x}, t) = \bar{\rho}_\gamma(t) \int \frac{d^2p}{4\pi} J(\vec{x}, \hat{p}, t)\hat{p}_i\hat{p}_j, \tag{12}$$

$$\delta T^{0}_{\gamma\,j}(\vec{x}, t) = a(t)\bar{\rho}_\gamma(t) \int \frac{d^2p}{4\pi} J(\vec{x}, \hat{p}, t)\hat{p}_j, \tag{13}$$

$$\delta T^{0}_{\gamma\,0}(\vec{x}, t) = -\bar{\rho}_\gamma(t) \int \frac{d^2p}{4\pi} J(\vec{x}, \hat{p}, t). \tag{14}$$

$$\tag{15}$$

The important point here is that by writing things this way, we have eliminated any direct dependence of the stress-tensor on the *magnitude* p of the photon momentum. This is important because photons scatter only via Thomson scattering. While Thomson scattering changes the photon *direction*, it does not change its energy.

Now consider a particular Fourier mode. The Boltzmann equation for the Fourier amplitude for the intensity is then

$$\begin{aligned}\frac{\partial J(\vec{q}, \hat{p}, t)}{\partial t} &+ i\frac{\hat{p}\cdot\vec{q}}{a(t)} J(\vec{q}, \hat{p}, t) + 2\alpha(\vec{q}) \left[\dot{A}_q(t) - (\vec{q}\cdot\hat{p})^2 \dot{B}_q(t)\right] = \\ &-\omega_c(t) J(\vec{q}, \hat{p}, t) + \omega_c(t) \int \frac{d^2\hat{p}_1}{4\pi} J(\vec{q}, \hat{p}_1, t) + \frac{4\omega_c(t)}{a(t)} \hat{p}\cdot\delta\vec{u}_B(\vec{q}, t),\end{aligned} \tag{16}$$

where $\delta\vec{u}_B(\vec{q}, t) = i\alpha(\vec{q})\vec{q}\delta u_B(t)$. Remember, J is related to the phase-space density of photons with direction $\hat{p}$. The first two terms on the left-hand side arise even if there are no perturbations nor scattering; they simply constitute the Lagrangian time derivative, $dJ/dt = (\partial/\partial t + \hat{\vec{v}} \cdot \nabla)J$, where $\vec{v}$ is the photon velocity. The last term on the left-hand side describes the deflection of the photon trajectory in the perturbed spacetime. The first term on the right-hand side describes the removal of photons from this region of phase-space via scattering with a collision rate $\omega_c(t)$. The second term on the right-hand side describes the scattering of photons *into* this region of phase space. The last term describes the effects of scattering from baryons if there is a baryon-photon relative velocity.

We then note that the direction $\hat{q}$ of the wavevector $\vec{q}$ arises only through a dot product with $\hat{p}$, the photon direction. We can thus write $J(\vec{q}, \hat{p}, t) = \alpha(\vec{q})\Delta_T(q, \mu, t)$, with $\mu \equiv \hat{q}\cdot\hat{p}$ and define a "source function" Φ through $\int (d^2\hat{p}/(4\pi)) J(\vec{q}, \hat{p}, t) = 3\alpha(\vec{q})\Phi(q, t)$.

We then arrive at an integro-differential equation,

$$\dot{\Delta}_T(q, \mu, t) + i\frac{q\mu}{at(t)}\Delta_T(q, \mu, t) = -\omega_c(t)\Delta_T(q, \mu, t) - 2\dot{A}_q(t) + 2q^2\mu^2\dot{B}(t) + 3\omega_c(t)\Phi(q, t) + 4iq\mu\omega_c(t)\delta u_B(t), \tag{17}$$

for each value of q. The reason that its an *integro*-differential equation is that there is an integral over μ in the definition of $\Phi(q,t)$. This equation is usually solved numerically by expanding the μ dependence of $\Delta_T(q,\mu,t)$ in Legendre polynomials,

$$\Delta_T(q,\mu,t) = \sum_{\ell=0}^{\infty} i^{-\ell}(2\ell+1)P_\ell(\mu)\Delta_{T,\ell}(q,t), \tag{18}$$

Then, using the orthonormality of the Legendre polynomials, the Boltzmann equation can be re-written,

$$\dot{\Delta}_{T,\ell}+\frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta_{T,\ell+1}-\ell\Delta_{T,\ell-1}\right] = -2\dot{A}_q\delta_{\ell 0}+2q^2\left(\frac{\delta_{\ell 0}}{3}-\frac{2\delta_{\ell 2}}{15}\right)-\omega_c\Delta_{T,\ell}+3\Phi\omega_c\delta_{\ell 0}-\frac{4}{3}q\omega_c\delta u_B\delta_{\ell 1}, \tag{19}$$

with $\Phi = (2\Delta_{T,0}-\Delta_{T,2})/6$. In terms of these $\Delta_{T\ell}$ moments, the photon energy-density perturbation is $\delta\rho_\gamma = \bar{\rho}_\gamma\Delta_{T,0}$; the pressure perturbation is $\delta p_\gamma = (\bar{\rho}_\gamma/3)(\Delta_{T,0}+\Delta_{T,2})$; the photon velocity is $qu_\gamma = -(3/4)\Delta_{T,1}$; and the photon shear stress is $q^2\pi_{gamma} = \bar{\rho}_\gamma\Delta_{T,2}$.

In practice, what we have done is an oversimplification. In practice, Thomson scattering of photons depends on their polarization, and so the intensity J is replaced by an intensity tensor J_{ij}; and there is an additional set of moments for $\Delta_{P,\ell}$ that must be included to describe the polarization fluctuations.

There is then an analogous set of Boltzmann equations for the evolution of the neutrino phase-space distribution. That set of equations is a bit simpler because the neutrinos are assumed to be collisionless. If, however, neutrinos have masses, then it is no longer true that the Boltzmann equations depend only on the neutrino propagation direction; they also can depend on the magnitude of the neutrino momentum, and this makes for a very significant additional complication.

We thus have so far derived a simple equation for the evolution of the dark-matter density perturbation; a relatively simple equation for the evolution of the baryon-density perturbation; and then an infinite sequence (usually cut off at some sufficiently high ℓ) of differential equations for the photons. There are also an analogous set of equations for the neutrinos. The last step are two Einstein equations for the evolution of the perturbations $A(t)$ and $B(t)$. The first of these can be written,

$$\frac{\partial}{\partial t}\left[a^2\psi_q\right] = -4\pi G a^2\left(\delta\rho+3\delta p - q^2\pi\right), \tag{20}$$

in terms of $\psi \equiv (3\dot{A}-q^2\dot{B})/2$. Here, the density perturbation $\delta\rho$ receive contributions from photons, dark matter, baryons, and neutrinos, and the pressure δp and anisotropic stress ϕ receive contributions from the photons and neutrinos. The second Einstein equation can be written,

$$\dot{A}_q = 8\pi G\left[\frac{4}{3}\bar{\rho}_\gamma\delta u_\gamma+\frac{4}{3}\bar{\rho}_\nu\delta u_\nu+\bar{\rho}_B\delta u_B\right], \tag{21}$$

recalling that in synchronous gauge, $\delta u_D = 0$.

We then must integrate coupled differential equations for the dark-matter density and another for the baryon density; two for the scalar potentials $A(t)$ and $B(t)$; a huge number for the photon moments $\Delta_{T,\ell}$ and $\Delta_{P,\ell}$; and a similar set for the neutrinos. The initial conditions at $t\to 0$ are specified by the early-Universe theory. For example, in the simplest models the perturbations are taken to be adiabatic. This fixes the initial fractional density and pressure perturbations to be equal and simply related to the initial curvature perturbation, which fixes $A(t)$ and $B(t)$ in the $t \to 0$ limit. There is then a simple prescription for the intial photon and neutrino phase-space densities as well.

2 The hydrodynamic limit

The precision of current CMB measurements requires that the complete system of equations, up to $\ell \sim 3000$ for the photons, be evolved numerically. Still, qualitative features and approximate results can be understood with analytic simplifications.

We'll begin by considering early times ($z \gg 3000$), well before recombination, and when the Universe is radiation dominated ($\bar{\rho}_M \ll \bar{\rho}_R$), with $a(t) \propto t^{1/2}$ and $H = (2t)^{-1}$. Before recombination, there are plenty of free electrons for photons to Thomson scatter from, and so the photon mean-free path is tiny. Quantitatively, this is described by taking the limit $\omega_c(t) \to \infty$ in the photon Boltzmann equations. As a result, all the photon multipole moments with $\ell \geq 2$ can be set to zero. We can also set the photon and baryon velocities to be equal (i.e., the photons and baryons make up a single photon-baryon fluid). Neutrinos are a bit more complicated because they free-stream, but we'll consider superhorizon modes ($q/a \ll H$) in which case the neutrino hierarchy can also be truncated at $\ell = 1$. We then define $\delta_\alpha \equiv \delta\rho_\alpha/(\bar{\rho}_\alpha + \bar{p}_\alpha)$.

The synchronous-gauge gravitational field equation for a given Fourier mode then becomes

$$\frac{d}{dt}(t\psi) = -4\pi G t \left(\bar{\rho}_D \delta_D + \bar{\rho}_B \delta_B + \frac{8}{3}\bar{\rho}_\gamma \delta_\gamma + \frac{8}{3}\bar{\rho}_\nu \delta_\nu \right). \tag{22}$$

The photon-baryon fluid equations are

$$\dot{\delta}_\gamma = \dot{\delta}_B = -\psi_q + (q^2/a^2)\delta_\gamma. \tag{23}$$

This equation simply tells us that the photon-baryon fluid falls in the gravitational field (the first term) but the growth of perturbations is also affected by the fact that the photon-baryon fluid has a non-zero pressure. The equation of motion for the dark-matter perturbation looks the same, but without the pressure term:

$$\dot{\delta}_D = -\psi_q. \tag{24}$$

The neutrino equation looks like the radiation equation:

$$\dot{\delta}_\nu = -\psi_q + (q^2/a^2)\delta_\nu. \tag{25}$$

The continuity equation for the photons is

$$\frac{d}{dt}\left(\frac{\delta u_\gamma}{\sqrt{t}}\right) = -\frac{1}{3\sqrt{t}}\delta_\gamma, \tag{26}$$

and there is a similar equation for the neutrinos.

2.1 The growing adiabatic mode

Motivated by inflation, we surmise that at sufficiently early times we can set all the δ_α equal and also all the δu_α—i.e., adiabatic initial conditions. At sufficiently early times, the dark-matter and baryon densities contribute negligibly in the Einstein equation, and at sufficiently early times we can neglect the q^2/a^2 terms. The resulting equations can then be combined to a single second-order differential equation for all the δ_α:

$$\frac{d}{dt}\left(t\frac{d}{dt}\delta\right) - \frac{1}{t}\delta = 0. \tag{27}$$

This is a second-order equation and so has two solutions. The decaying solution can is irrelevant not only because it does not survive to late times, but also because it is associated with zero curvature, $\mathcal{R} = 0$. The growing mode, which is the one we usually refer to as *the* adiabatic mode, has solution,

$$\delta = \frac{q^2 t^2 \mathcal{R}^0}{a^2}, \quad \psi = -\frac{t q^2 \mathcal{R}^0}{a^2}, \quad \delta_\gamma = \delta_\nu = -\frac{2t^3 q^2 \mathcal{R}^0}{9a^2}, \tag{28}$$

where here the superscript 0 designates the (constant) value of the curvature on superhorizon scales, $q^2/a^2 \ll H$.

Most generally, in the absence of the inflation-inspired adiabatic condition, there are eight independent initial conditions that must be specified for each Fourier mode and thus eight independent modes, four of which will be growing relative four others that decay. As one example, there is a mode with

$$\delta_D = \frac{\epsilon \bar\rho B}{\bar\rho_B + \bar\rho_D}, \quad \delta_B = \frac{\epsilon \bar\rho D}{\bar\rho_B + \bar\rho_D}, \tag{29}$$

and $\psi = \delta_\gamma = \delta_\nu = \delta u_\gamma = \delta_\nu = 0$. This mode has $\mathcal{R} = 0$ and is thus called an "isocurvature" mode. In this mode, the baryon and dark-matter perturbations are chosen so that the initial total-matter perturbation and so that the curvature perturbation is zero In the recent literature, this is thus called a *compensated* isocurvature perturbation. More generally, the term "isocurvature" is applied often in the literature to any non-adiabatic mode, even those which (strictly speaking) may have nonzero curvature. Its just sloppy terminology.

2.2 Extension to later times

We now consider the evolution of modes to later times, when the ratio $R \equiv 3\bar\rho_B/4\bar\rho_R$ is not necessarily small, and to when the perturbations re-enter the horizon, so that q/a is no longer necessarily negligible compared with H. The equations are thus generalized to

$$\frac{d}{dt}\left(a^2\psi\right) = -4\pi G a^2 \left[\bar\rho_D \delta_D + \left(\bar\rho_B + \frac{8}{3}\bar\rho_\gamma\right) + \frac{8}{3}\bar\rho_\nu \delta_\nu\right], \tag{30}$$

$$\dot\delta_\gamma - (q^2/a^2)\delta u_\gamma = -\psi, \tag{31}$$

and similarly for δ_ν,

$$\dot\delta_D = -\psi, \tag{32}$$

$$\frac{d}{dt}\left(\frac{(1+R)\delta u_\gamma}{a}\right) = -\frac{1}{3a}\delta_\gamma, \tag{33}$$

$$\frac{d}{dt}\left(\frac{\delta u_\nu}{a}\right) = -\frac{1}{3a}\delta_\nu. \tag{34}$$

The appearance of R in the second to last, but not the last, equation occurs because the photons are still part of the baryon-photon fluid (we are still working in the regime $z \gtrsim 1100$ where the photons and baryons are tightly coupled in a single photon-baryon fluid).

These equations cannot be solved analytically in full generality. Instead, we will consider the limits of long wavelengths $q \ll q_{\rm eq}$ and short wavelengths $q \gg q_{\rm eq}$, where the wavenumber $q_{\rm eq}$ that separate the two regimes is that which enters the horizon at matter-radiation equality; i.e., when $q_{\rm eq}/a_{\rm eq} = H_{\rm eq}$, or $R = 3/4$. In other words, here long-wavelength modes are those that enter the horizon later, during matter domination, and *vice versa* for short-wavelength modes, which enter the horizon during radiation domination. This critical wavenumber corresponds to a physical wavelength today of $\lambda_0 \equiv 2\pi/(q_{\rm eq}/a_0) = 850(\Omega_m h^2/0.1)^{-1}$ Mpc. This distance scale is huge compared to those of galaxies and galaxy clusters, and so most of the distance scales relevant for galaxy surveys entered the horizon during radiation domination. In the CMB, this critical wavenumber corresponds to a multipole moment $\ell \sim 140$. Thus, modes that affect the CMB power spectrum at $\ell \lesssim 140$ entered the horizon during matter domination, and those that influence at $\ell \gtrsim 140$ entered the horizon during radiation domination.

2.2.1 Long-wavelength modes

The solution for long-wavelength modes is obtained in the following way: First, the equations can be solved for superhorizon perturbations ($q/a \ll H$), as was done above, but now including the R dependence (before we took $R \ll 1$). Technically, an analytic solution can be found by replacing the independent variable t with the scale factor $y \equiv \bar\rho_M/\bar\rho_R$ (scaled to unity at matter-radiation equality). The equations can then also be solved for general $q/(aH)$, assuming matter domination. In this approximation, one assumes that the radiation density is negligible compared with the matter density, $a \propto t^{2/3}$, and $H = 2/3t$, and the total-density perturbation is dominated by the baryon and dark-matter perturbations. One then matches the two solutions near horizon crossing (for the modes that enter the horizon during MD, as we are considering here) to find solutions valid at all times. The results are straightforward but lengthy—you can find them in Weinberg's book, for example. Here we simply highlight the main results. To be clear, we are considering here solutions only for adiabatic initial conditions.

The result is that for long-wavelength modes during matter domination, the density perturbation is

$$\delta_D = \frac{9q^2t^2\mathcal{R}^0}{10\,a^2}, \tag{35}$$

and the metric perturbation variable is

$$\psi = -\frac{3q^2t\mathcal{R}^0}{5\,a^2}. \tag{36}$$

Note that since $a \propto t^{2/3}$ during MD, the density perturbation grows as $\delta_D \propto t^{2/3}$, which recovers (reassuringly) the result we obtained from our earlier Newtonian analysis for the growth of density perturbations in the MD era. Recalling also that the Newtonian potential in the Newtonian treatment is (from the Poisson equation) $\phi = -4\pi G\bar{\rho}_D\delta_D$, we identify $\phi = -(3/5)\mathcal{R}^0$. We have thus related the curvature amplitude that results from inflation to the initial conditions for density perturbations (for long-wavelength modes) when they re-enter the horizon.

The growth of perturbations to the baryon density is a bit more subtle. Once a given mode enters the horizon, gravitational infall amplifies the growth of dark-matter perturbations, as seen above and as seen in our earlier Newtonian treatment. The baryons, however, are tightly coupled to the photons until recombination at $z \simeq 1100$, which happens a bit after the Universe transitions to matter domination, at $z \simeq 3000$. For modes with wavelengths so long that they re-enter the horizon after recombination, the dark matter behaves pretty much like the baryons. However, for those modes that enter the horizon after matter domination but before recombination, there is a brief period, between horizon crossing and recombination, when things are a bit more complicated. For these modes, the pressure in the photon-baryon fluid opposes the growth of density perturbations. Baryon perturbations on these scales are thus suppressed relative to the dark-matter perturbations. This becomes a far bigger effect for small-wavelength modes. We therefore now focus on small-wavelength modes to avoid a significant investment of complication for a not-too-dramatic conclusion.

2.2.2 Short-wavelength modes

The evolution of short-wavelength modes—those that enter the horizon during radiation domination—is a lot more complicated for several reasons: First and foremost, the photon-baryon fluid has a considerable pressure, and so perturbations in the photon-baryon fluid wind up oscillating (as acoustic waves), rather than simply growing monotonically, once they enter the horizon. The amplitude of the metric perturbation, and thus the dark matter, which feels only the gravitational field, does not oscillate as much. While there is a small oscillatory component, the potential and dark-matter perturbations continue to grow. Since, however, the energy density is dominated by the radiation, which remains smoother due to its pressure, the growth of dark-matter and potential perturbations is far slower than it is during superhorizon evolution of perturbations or for subhorizon perturbations, during matter domination. There is finally dissipation on very small scales.

In more technical terms, the system of dark-matter, neutrino, and photon-baryon perturbations constitutes a sixth-order system and is described most generally by six modes, which can be decomposed into four fast modes (that evolve on timsecales much smaller than the Hubble time) and two slow modes (that evolve on roughly a Hubble time). The slow modes are important for the evolution of radiation perturbations (CMB) and experience dissipation, while the fast are more relevant for dark matter.

2.3 Radiation-dominated era

All short-wavelength modes enter the horizon during radiation domination, when $a \propto t^{1/2}$ and $H = 1/2t$ and when $\rho_R \gg \rho_D$. As Weinberg shows, the solution for the baryon-photon perturbation is,

$$\delta = 3\mathcal{R}\left(\frac{2}{\Theta}\sin\Theta - \left(1-\frac{2}{\Theta^2}\right)\cos\Theta - \frac{2}{\Theta^2}\right), \tag{37}$$

where $\Theta = 2qt/(\sqrt{3}a)$; there are similar expression for the other perturbation variables. Of more importance than the precise analytic form is simply the observation that the solution is oscillatory—these are the acoustic waves discussed above. The expressions for the dark matter and potential involve integrals over oscillatory functions, and so are thus evolve more smoothly with time.

Now consider modes on scales sufficiently small that they are well within the horizon and consider the evolution of the photon-baryon fluid. In this case (as Weinberg shows explicitly, and as our Newtonian discussion of pressure gradients in connection with the Jeans instability suggests), the evolution of the photon-baryon fluid is that of a smooth gas in an adiabatically expanding box. The equations for the perturbations in the photon-baryon fluid then reduce to a wave equation with sound speed $v_s = [3(1+R)]^{-1/2}$ (that evolves slowly with time) in an expanding Universe. For a wave of physical wavenumber q/a, the oscillation frequency is $v_s(q/a)$. If this frequency is large compared with the expansion rate H, then the change in the wavenumber and frequency with time can be approximated as adiabatic and the wave equation solved with a WKB approximation:

$$\delta_\gamma \simeq (1+R)^{-1/4} \exp\left[\pm iq \int_0^t \frac{dt}{a\sqrt{3(1+R)}}\right]. \tag{38}$$

On the very smallest scales, the finite value of the photon mean-free path introduces a finite viscosity which leads to dissipation (into heat) of the acoustic waves. This is known as *Silk damping*. The result is to damp the perturbations by a factor $\exp\left[-\int_0^t \Gamma\, dt\right]$, where

$$\Gamma(t) = \frac{q^2 t_\gamma}{6a^2(1+R)}\left(\frac{16}{15} + \frac{R^2}{1+R}\right), \tag{39}$$

where $t_\gamma = (\sigma_T n_e)^{-1}$ is the mean-free time for photon scattering. The derivation of this result is straightforward but pretty complicated (it requires, for example, that you take into account the polarization dependence of Thomson scattering), but it is easy to understand in an order-of-magnitude sense: Γ^{-1} is simply the time for a photon of mean-free path ct_γ to diffuse a distance $(q/a)^{-1}$. The photon mean-free path is extremely small at early times. However, when recombination begins, the free-electron abundance is rapidly reduced, and the photon mean-free path grows very quickly. Silk damping thus very effectively damps radiation-density perturbations on small scales just before CMB photons last scatter. It thus leads to a signicant damping of the CMB power spectrum at large ℓ (as we will see). However, Silk damping has little effect on the matter power spectrum.

OK. So now we've seen that deep within the horizon the radiation perturbations oscillate more or less as they would in a smooth expanding background spacetime. Now let's think about what the dark matter and density perturbations are doing for modes that are deep within the horizon. Above we saw that the slowness of the cosmological evolution allowed us to decouple the fast oscillations in the photon-baryon fluid from the (relatively) slow evolution of the Universe. The converse is now true for the dark matter. Although the photon-baryon fluid has fluctuations, their effect on the dark matter is smoothed out after averaging over a number of oscillation cycles. If we assume that the photon-baryon fluid remains smooth, then the equation of motion for the evolution of the perturbation to the dark-matter density reduces to that we derived in the Newtonian limit: in the current notation, we can write it as [Eq. (6.4.33) in Weinberg],

$$\ddot{\delta}_D + 2\frac{\dot{a}}{a} - 4\pi G\bar{\rho}_D\delta_D = 0. \tag{40}$$

However, now the scale factor is not necessarily $a \propto t^{2/3}$ as we assumed for matter domination but $a \propto t^{1/2}$ at early times and more generally some interpolation between the two. Eq. (40) can be solved by replacing the independent variable t with $y \equiv a/a_{\rm eq}$. The precise solutions depend on the baryon density, but in the limit that $\rho_B \ll \rho_D$, the solution is a linear combination, $\delta_D = c_1\delta_D^{(1)} + c_2\delta_D^{(2)}$ of the two linearly independent solutions,

$$\delta_D^{(1)} = 1 + \frac{3y}{2}, \tag{41}$$

$$\delta_D^{(2)} = \left(1 + \frac{3y}{2}\right)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y}. \tag{42}$$

The first of these recovers the late-time ($y \to \infty$, MD) growing mode ($\delta \propto t^{2/3}$) we found in the Newtonian analysis—this will be the mode that determines the density perturbation at late times. To fix the amplitude of this mode, we must choose the coefficients c_1 and c_2 so that the solution (and its first derivative) match the solution obtained for the evolution early in the radiation-dominated era, the solution for δ_D that corresponds to that for δ_γ in Eq. (37). This turns out to be

$$\delta_D = 6\mathcal{R}^0\left(-\frac{1}{2} + \gamma + \ln\Theta\right), \tag{43}$$

where $\gamma = 0.5772$ is the Euler constant. As a result, the dark-matter density perturbation becomes, deep in the matter-dominated era,

$$\delta_D = \frac{9{\mathcal{R}_{\rm II}}^0 a}{a_{\rm eq}}\left[-\frac{7}{2} + \gamma + \ln\left(\frac{4\kappa}{\sqrt{3}}\right)\right]. \tag{44}$$

where $\kappa = \sqrt{2}q/q_{\rm eq}$.

2.4 The matter transfer function

In our discussion of Newtonian perturbations, we described the matter power spectrum as $P(q) = P_{\rm primordial}(q)[T(q)]^2$, in terms of the nearly scale-invariant primordial power spectrum from inflation and a transfer function $T(q)$ that is approximated in the long- and short-wavelength limits by

$$T(q) \simeq \begin{cases} 1, & q \lesssim q_{\rm eq}, \\ (q/q_{\rm eq})^{-2}, & q \gtrsim q_{\rm eq}, \end{cases} \tag{45}$$

We are now in a position to see how this arises. We found in Eq. (35) that for long-wavelength modes, which enter the horizon during matter domination, the density-perturbation is

$$\delta_D = \frac{9q^2t^2\mathcal{R}^0}{10\,a^2}T(q), \tag{46}$$

with $T(q) \to 1$, while for short-wavelength modes, which enter the horizon during radiation domination, it takes the same form but [cf. Eq. (44)] with

$$T(q) \to \frac{45\, q_{\rm eq}^2}{4q^2}\left[-\frac{7}{2}+\gamma+\ln\left(\frac{4\sqrt{2}q}{\sqrt{3}q_{\rm eq}}\right)\right]. \qquad (47)$$

which does indeed recover the transfer-function shape discussed above.

The amplitude of small-scale perturbations is suppressed because the growth of subhorizon perturbations is logarithmic during the radiation-dominated era. When perturbations are super-horizon, their amplitude grows as $\delta_D \propto t$ during RD and as $\delta_D \propto t^{2/3}$ during MD. For long-wavelength modes, which enter the horizon during MD, the growth continues as $\delta \propto t^{2/3}$ even when they become subhorizon. For short-wavelength modes, the growth gets slowed to logarithmic, $\delta_D \propto \ln t$, when they enter the horizon during radiation domination (qualitatively, because perturbations to the dominant radiation density do not grown) before then resuming a $\delta_D \propto t^{2/3}$ during matter domination. This stunting of the growth begins at the time t_c when the mode enters the horizon, given by $q/a \sim 1/t_c$. Since $(a/a_{\rm eq}) \sim (t/t_{\rm eq})^{1/2}$ this is when $t \sim t_{\rm eq}(q_{\rm eq}/q)^2$. The MD growth factor $t_{\rm eq}/t \sim (q/q_{\rm eq})^2$ is thus replaced by $\ln q$.

These arguments give us the correct qualitative behavior for the matter power spectrum in the two limiting regimes, and they also give the correct limits if (a) the baryon density is negligible; (b) neutrino perturbations are neglected; and (c) to the extent that the slow/fast decomposition of the small-wavelength behavior is well approximated by the WKB solution. In practice, the precision of current CMB and galaxy-survey measurements require that the complete set of equations be evolved numerically not only to get the proper interpolation, but also to get the effects of baryons and neutrinos right in the small-wavelength limit.

There are also baryon acoustic oscillations in the matter power spectrum that we can understand. As mentioned above, we assumed in the evolution of the matter power spectrum that the baryon density was negligible. In practice, though, roughly 1/6 of the nonrelativistic matter in the Universe is baryons which, before recombination, are tightly coupled to the photons. Since the baryons move with the photons before recombination, they experience acoustic oscillations. To get a qualitative understanding, consider Eq. (37) which although strictly speaking valid only in radiation domination, remains qualitatively similar in MD. This equation oscillates with $\Theta \propto q$. Thus, although it oscillates for fixed q with time, it also oscillates, at some fixed time t, with q. Thus, at recombination, when the tight coupling of baryons to photons ceases, the baryon-density transfer function oscillates sinusoidally with wavenumber q with period $\sim t_{\rm rec}^{-1}$. The total matter-density perturbation at recombination is thus (schematically—there are some technical complications in the detailed calculation) $\delta_M = \bar{\rho}_D\delta_D + \bar{\rho}_B\delta_B$. Although δ_D varies monotonically with wavenumber q, the baryon perturbation has an oscillatory component. There is thus a small oscillatory component superimposed on the otherwise smooth power spectrum that would arise without baryons.

The growth of perturbations and baryon acoustic oscillations can be thought of in a configuration-space (rather than Fourier-space) pictures. Consider a smooth Universe with an initial spherical adiabatic tophat perturbation superimposed. The dark matter and baryon-photon fluid in this initial overdensity will at early times both fall similarly in a manner described by the early evolution of a spherical tophat we discussed earlier. At some point, when the photon-baryon fluid becomes sufficiently overdense, the pressure gradient will overcome the gravitational attraction of the overdensity, the photon-baryon fluid will reach a point of maximum compression and then bounce back, sending a spherical shock wave, which propagates at the speed of sound, back into the photon-baryon fluid in the surrounding medium. This spherically-symmetric fractional photon-baryon density perturbation $\delta(\vec{x}, t)$ at time t can be Fourier transformed as

$$\delta(\vec{x}, t) = \int \frac{k^2\, d}{2\pi^2} \delta_{\vec{q}}(t) \frac{\sin kr}{kr}, \tag{48}$$

where $\delta_{\vec{q}}$ satisfy the Fourier-space equations we derived above. At recombination, when the CMB snapshot we see is created, the shock front has propagated a distance $\sim v_s t_{\rm rec}$. This relatively sharp shock front then winds up having oscillatory features in Fourier space. Thus, the baryon acoustic oscillations in the matter power spectrum represent a Fourier-space description of these shock fronts that propagate at the speed of sound.