Week 1: Cosmography

January 9, 2012

1 Distance and Mass Scales

In these lectures we will develop a quantitative model to describe the structure of the observable Universe as a whole. To begin, let us review the relevant physical scales. An Earth mass is $6 \times 10^{27}\,\mathrm{g} \simeq 3 \times 10^{51}\,\mathrm{GeV}/c^2 \simeq 3 \times 10^{-6}\,M_\odot$. Its radius is 6.4×10^8 cm. It spins around the Sun, which has a mass $M_\odot = 2 \times 10^{33}\,\mathrm{g} \simeq 10^{57}\,\mathrm{GeV}/c^2$ and radius $R_\odot \simeq 7 \times 10^{10}\,\mathrm{cm}$. The Earth-Sun distance is $1\,\mathrm{AU} = 1.5 \times 10^{13}$ cm. The mass of Jupiter, the largest planet is $\sim 10^{-3}\,M_\odot$.

The characteristic spacing between stars is roughly a parsec ($1\,\mathrm{pc} = 3 \times 10^{18}$ cm), and the Sun spins around in our galaxy, the Milky Way, a gravitationally bound collection of several billion stars. The stellar mass of the Milky Way is $\simeq 3 \times 10^{10}\,M_\odot$ (that does not include the dark matter, which we may get to later), and the radius of the stellar disk in the Milky Way is ~ 10 kpc. As discovered by Hubble and others early in the 20th century, the Milky Way is nothing special; it's just one of billions of galaxies that we can see, and the typical spacing between these galaxies is $\sim$Mpc.

2 The expansion

Hubble discovered in the late 1920s that every galaxy moves away from us at a recessional velocity $v \equiv cz$ (where c is the speed of light and z is the "redshift") proportional to its distance d_L: i.e., $v = H_0 d_\mathrm{L}$, where $H_0 = 100\,h$ km/sec/Mpc is the Hubble parameter, and current measurements give us $h \simeq 0.7$ to roughly 10%. It is also observed that on the largest observable scales (100s to 1000s of Mpc) the galaxy distribution is roughly isotropic; we see roughly the same distribution of galaxies in every direction we look—later this quarter we will quantify this more precisely. If we then apply the Copernican principle, which states that we do not occupy a preferred position in the Universe, then it follows that any observer in any other galaxy should also see an isotropic distribution of galaxies. This can only be accommodated if the Universe is, in addition to being isotropic, homogeneous.

The Hubble law $v = H_0 d$ and the assumption of homogeneity can only be reconciled if the relative velocity between any two galaxies in the Universe is proportional to the distance between them.

And this is reconciled if the Universe undergoes self-similar expansion in the following way. Suppose that the position $\vec{r}_i(t)$ of each galaxy (labelled by a subscript i) is written as a function of time t as $\vec{r}_i(t) = a(t)\vec{x}_i$, where $\vec{x}_i$ is the *comoving position* of galaxy i, which remains fixed for all time t, and $a(t)$ is a *scale factor*, which describes the time dependence of the spacing between galaxies. The relative velocity between any two galaxies i and j separated (physically) by $\vec{r}_i(t) - \vec{r}_j(t)$ becomes $\vec{v} = \dot{\vec{r}}_i \; \dot{\vec{r}}_j = (\vec{x}_i - \vec{x}_j)\dot{a} = (\vec{r}_i - \vec{r}_j)(\dot{a}/a)$ (where the overdot denotes derivative with respect to time), from which we infer that the Hubble parameter and scale factor are related by $H_0 = \dot{a}/a$. Note also that as long as $|v| \ll c$, the redshift z describes the Doppler shift between the wavelength λ_e of light emitted by galaxy "e" and the wavelength λ_o observed by galaxy "o": $z \equiv v/c = H_0\vec{r}/c = (\lambda_{\rm o}/\lambda_{\rm e}) - 1$.

3 Expansion time and distance

If we divide the galaxy separation by the relative velocity, we obtain a characteristic expansion time $t \sim r/v = H_0^{-1} \simeq 1.5 \times 10^{10}$ yr which, we will see, is quite close to the age of the Universe. Quite remarkably, this is quite close to the ages of the oldest stars, even though the timescale for stellar evolution is determined by an amalgamation of gravity and nuclear and atomic physics, and should thus should *a priori* have nothing to do with the expansion rate H_0 of the Universe. The distance that light can travel in this time is $l \sim cH_0^{-1} \simeq 5000\,{\rm Mpc} \simeq 1.5 \times 10^{28}\,{\rm cm}$, which we will later see is numerically quite close to the size of the observable Universe. Also, in order of magnitude, the volume of the Universe should be roughly $V = (4/3)\pi l^3 \sim 3 \times 10^{11}\,{\rm Mpc}^3$, and should thus contain roughly several hundred billion galaxies.

4 Cosmography of the Universe

Above we introduced the notion of comoving coordinates, a coordinate system in which galaxy positions remain fixed, and in which the expansion is accounted for by a scale factor $a(t)$. These coordinates are convenient because galaxy positions are fixed. However, suppose we want to be able to discuss things like the physical distance to galaxies, the time it takes light signals to propagate between galaxies, or the relative velocities between galaxies, especially at large distances where our earlier analysis seems to suggest superluminal motions. For these reasons, we must develop a more sophisticated definition of what we mean by comoving and physical coordinates.

This is done by introducing the *metric* (or really, the *line element*) for the expanding Universe. A metric is something you learn about very early in any general relativity class. However, you don't need the full machinery of general relativity to understand what a metric is.

The simplest example of a metric is that for special relativity, the Minkowski spacetime,

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2. \tag{1}$$

What this equation says is simply that if we consider two spacetime points $(t, \vec{x})$ separated by $(dt, d\vec{x})$, then the spacetime separation between those points is ds. You may also recall that light rays propagate along "null geodesics," trajectories which have $ds = 0$ (i.e., they move with velocities equal to the speed of light c, which I will usually set equal to 1), while massive particles travel along trajectories in spacetime with $ds > 0$.

It does not require too much of an intuitive leap to then guess that if x, y, and z are now comoving coordinates in an expanding homogeneous Universe, then the differentials dx, dy, and dz, which above are physical separations, should now be replaced by $a(t)dx$, $a(t)dy$, and $a(t)dz$; i.e., the metric for an expanding Universe should be

$$ds^2 = dt^2 - [a(t)]^2(dx^2 + dy^2 + dz^2). \tag{2}$$

Remarkably enough, this turns out to be entirely correct; this is the *Friedman-Robertson-Walker* metric for a spatially flat Universe.

Before congratulating ourselves on this success and moving on, we should re-derive this result in a slightly more sophisticated way, and also realize that this metric, although acceptable, is not fully general. That is, there may be other metrics for a homogeneous, isotropic, and uniformly expanding Universe, like our own. The one we derived, for a *flat* Universe, is only one of three possibilities, the other two being the *closed* and *open* universes.

To begin and develop intuition, it is easiest to start with an analogue with two (rather than three) spatial dimensions. We need a metric for a homogeneous and isotropic space. The simplest is for $\Re^2$: $dl^2 = dx^2 + dy^2$ in Cartesian coordinate (x, y), or in polar coordinate (r, θ)—defined by $x = r\cos\theta$ and $y = r\sin\theta$—$dl^2 = dr^2 + r^2 d\theta^2$. This space has zero curvature (all components of the Riemann curvature tensor vanish). The differential volume element is $dV = dx\,dy$, and the total volume is infinite.

The second possibility is the two-sphere S^2. The surface of a two-sphere is homogeneous (every point has the same curvature) and isotropic (the surface of the sphere looks the same in every direction you look from any given point on the sphere). (Note that the "volume" (i.e., surface area) is not infinite; however, since we only observe a finite volume of our own Universe, there is no guarantee that the volume beyond our observable Universe is infinite.) Consider a sphere of radius a. In coordinates (θ, ϕ) (polar and azimuthal angle), the metric is $dl^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$. The differential volume element is $dV = a^2\ \sin\theta\, d\theta\, d\phi$, and the total volume (integrating $\phi : 0 \to 2\pi$ and $\theta : 0 \to \pi$) is 4π.

There's another way we can write the metric for S^2. Embed S^2 in three spatial dimensions: $x_1^2 + x_2^2 + x_3^2 = a^2$. The distance between two points on the two-sphere is

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_1^2 + dx_2^2 + \frac{(x_1\,dx_1 + x_2\,dx_2)^2}{a^2 - x_1^2 - x_2^2}. \tag{3}$$

Then, defining new coordinates r and ϕ through $x_1 = ar\cos\phi$ and $x_2 = ar\sin\phi$, we find the metric,

$$dl^2 = a^2\left(\frac{dr^2}{1 - r^2} + r^2\,d\phi^2\right), \tag{4}$$

and note that $0 \le r \le 1$ and $0 \le \phi \le 2\pi$. Also note that these coordinates cover only half the sphere: either the northern half or the southern half. The two-sphere is a surface of constant positive curvature, and the Ricci scalar is everywhere equal to $2/a^2$.

There is a third possibility (in addition to $\Re^2$ and S^2): A surface of constant *negative* curvature can be obtained by replacing $a \to ia$: $x_1^2 + x_2^2 + x_3^2 = -a^2$. Then,

$$dl^2 = a^2 \left(\frac{dr^2}{1+r^2} + r^2\, d\phi^2 \right), \tag{5}$$

with $0 \le r < \infty$ or

$$dl^2 = a^2 \left(d\theta^2 + \sinh^2\theta\, d\phi^2 \right), \tag{6}$$

with $0 \le \theta < \infty$. This surface of constant negative curvature ($R = -2/a^2$) cannot be embedded in three spatial dimensions. The differential volume element is $dV = a^2 \sinh\theta\, d\theta\, d\phi$, and the total volume is infinite.

So we have found three homogeneous 2-dimensional metrics that can be written as

$$dl^2 = a^2 \left(\frac{dr^2}{1-kr^2} + r^2\, d\phi^2 \right), \tag{7}$$

where $k = 1$ for the two-sphere (a "closed" universe), $k = 0$ for the "flat" universe, and $k = -1$ for the "open" universe. It is easy to show (as you will do in the homework) that these are the only three possibilities for a homogeneous, isotropic two-dimensional universe.

Likewise, the metric for an isotropic and homogeneous universe with three spatial dimensions must be

$$dl^2 = a^2 \left(\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\, d\phi^2 \right), \tag{8}$$

where r is a radial coordinate, and θ and ϕ are the usual angular coordinates. For $k = 0$, this reduces simply to the Euclidean space $\Re^3$. For $k = 1$ (closed Universe), we can alternatively write,

$$dl^2 = a^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right], \tag{9}$$

where χ, an angle in the three-sphere ($0 \le \chi \le \pi$), plays the role of a radial coordinate, and for $k = -1$ (open),

$$dl^2 = a^2 \left[d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right], \tag{10}$$

and then $0 \le \chi \le \infty$.

What we have considered so far is the spatial metric. The most general *spacetime* metric for a homogeneous, isotropic universe is then the *Friedmann-Robertson-Walker (FRW) metric*,

$$ds^2 = dt^2 - [a(t)]^2 \left(\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\, d\phi^2 \right), \tag{11}$$

in (r, θ, ϕ) spatial coordinates, or

$$ds^2 = dt^2 - [a(t)]^2 \left[d\chi^2 + \begin{pmatrix} \sinh^2\chi \\ \chi^2 \\ \sin^2\chi \end{pmatrix} \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right], \tag{12}$$

in (χ, θ, ϕ) coordinates for (from top to bottom) $k = -1$, $k = 0$, and $k = 1$. Note that $a(t)$ has now been promoted to the scale factor, a function that increases with time t as the Universe expands. (Note that $a(t)$ is also the three-sphere radius for a closed Universe.) Keep in mind that although the flat (Euclidean, $k = 0$) universe has zero *spatial* curvature, it most generally has nonzero *spacetime* curvature.

Finally, while we're at it, we can also define a new time coordinate, the *conformal time* η by $d\eta = dt/a(t)$. Then,

$$ds^2 = a^2(\eta)\left[d\eta^2 - \frac{dr^2}{1 - kr^2} - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2\right]. \tag{13}$$

This is kind of nifty, as with this time coordinate, the metric is said to be *conformal* to the Minkowski metric; i.e., it is the same thing times some scale factor. With this metric, photons travel along the same coordinate trajectories as they would in Minkowski space. Keep in mind, though, that the conformal time is *not* the proper time—it is not the time measured by a comoving observer. That is still the original time coordinate t.

A note on normalizations: Textbooks (and certain teachers of Ay127...) are often careless about units and normalization in the FRW metric. My preference is to define r and χ to be dimensionless, in which case $a(t)$ has units of length (or time if you choose to leave out the c from the dt^2 term). Then, we can set the present day value of $a(t)$ to obtain the correct radius of curvature, which means we need

$$a(t_0) = \mathrm{c}H_0^{-1}\begin{cases}\sqrt{-1/\Omega_\mathrm{k}} & \text{if } k = +1\\ 1 & \text{if } k = 0\\ \sqrt{1/\Omega_\mathrm{k}} & \text{if } k = -1,\end{cases} \tag{14}$$

and $\Omega_\mathrm{k} = 1 - \Omega_\mathrm{m} - \Omega_\Lambda$ is the energy density in curvature (as we'll see later).

5 Equation of motion for the scale factor

In Newtonian gravity, the distribution of mass—the mass density $\rho(\vec{x})$—determines the gravitational potential ϕ through the Poisson equation, $\nabla^2\phi = 4\pi G\rho$. Particles then experience an acceleration $\nabla\phi$ in this gravitational field. In general relativity, the matter content of the Universe—i.e., its density $\rho(t)$ which is a function of time only in a homogeneous Universe—results in an equation of motion for the scale factor $a(t)$. This result is derived by writing down and solving the Einstein equation for the system, the general-relativitistic analogue of the Poisson equation that determines the relation between the components of the metric [in our case, $a(t)$ and k] and the matter density. If you've had general relativity, this is a straightforward exercise.

Fortunately, however, the result can be derived using only Newtonian physics. Suppose the Universe is homogeneous (i.e., has the same matter density $\rho(t)$ everywhere in space) and infinite (note that although the finite age of the Universe yields a finite *observable* volume, there is no reason why the spatial volume of the Universe cannot extend to infinity, well beyond the observable horizon). Now consider an arbitrary point in this Universe and draw a comoving sphere of radius x that will then have a physical radius $r(t) = a(t)r$. The volume of the Universe *outside* this sphere can be thought of a series of concentric spherical shells. Newton's theorem then tells us that the sphere in question does not feel any force due to any of the external concentric spherical shells, and we can therefore neglect the effect of the exterior on the interior. The dynamics of the spherical region of Universe in question is thus equivalent to that of a self-gravitating homogeneous spherical sphere. If we therefore consider a mass element dm at the surface of this imaginary sphere of radius a, then its acceleration relative to the center will be $\ddot{a} = -GM/a^2$, where $M = (4\pi/3)a^3\rho$ is the enclosed mass. In other words

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho. \tag{15}$$

Congratulations! You've just derived the second form of the *Friedmann equation*, which can be justified fully in general relativity.

The first form of the Friedmann equation can be derived alternatively using energy conservation. This imaginary sphere of radius a has a kinetic energy associated with the expansion and a potential energy due to the gravitational self-interaction. The gravitational binding energy for a homogeneous sphere of radius a is [cf., Ay123] $U = -(3/5)GM^2/a = -(16\,\pi^2/15)G\rho a^5$, where we have used $M = (4\pi/3)a^3\rho$ in the second step. Since a mass element at distance r from the center moves with velocity $v = Hr$, the kinetic energy of the expanding sphere is

$$T = 4\pi \int_0^a r^2\, dr\, \frac{1}{2}\rho H^2 r^2 = \frac{2\pi}{10}\rho H^2 a^5. \tag{16}$$

The total energy $E = T + U$ remains constant, and re-defining another constant

$$k \equiv -\frac{5}{2\pi}\frac{E}{\rho a^3}, \tag{17}$$

(noting that ρa^3 =constant) we arrive at the first (and more common) form of the Friedmann equation,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}. \tag{18}$$

This equation is of central importance in cosmology. It tells us how the time evolution of the scale factor is related to the matter density. By differentiating $\dot{a}$ with respect to time, you can show that it is equivalent to the second form of the Friedmann equation derived above. Although our derivation was Newtonian, the same result is obtained in the fully general-relativistic treatment. So why do we need general relativity? The Newtonian analysis ultimately breaks down for several reasons: e.g., (a) the gravitational potential one finds is infinite; and (b) since $v = H_0 d$, relative velocities become relativistic for distances $d \gtrsim cH_0^{-1}$ (and we see galaxies with redshifts $z > 1$). Moreover, the Newtonian analysis allows us to only consider *nonrelativistic matter*; i.e., matter in which the pressure is negligible compared with the energy density. In general relativity, pressures and stresses can also act as sources for the gravitational field, and general relativity will therefore be required to justify the Friedmann equation if radiation (which has a pressure one-third the energy density) or a cosmological constant (or vacuum energy, which has a pressure equal in magnitude but opposite in sign to the energy density) is dynamically important.

Finally (and perhaps most importantly) in the general-relativistic treatment, we find that the constant k that appears in the Friedmann equation is precisely the same as the constant k that appears in the FRW metric. This therefore relates the matter content to the geometry and to the ultimate fate of the Universe. Let us evaluate the Friedmann equation today—i.e., evaluating the Hubble parameter at its current value H_0 and the matter density ρ_0—and divide both sides of the

Friedmann equation by H_0^2. Then, define the density parameter $\Omega \equiv \rho_0/\rho_c$, where $\rho_c \equiv 3H_0^2/8\pi G$. Then, we have

$$\Omega_m = 1 + \frac{k}{a_0^2 H_0^2}. \tag{19}$$

We therefore see that the ratio of the matter density to the expansion rate (which can in principle be measured) is related to the geometry of the Universe. For $\Omega > 1$, $\Omega = 1$, $\Omega < 1$, the Universe is closed, open, or flat, respectively. Note, moreover, that the Friedmann equation is a first-order differential equation for the scale factor $a(t)$. We can use a simple energetic argument to determine how the solutions will behave, even without solving the differential equation. If the total energy $E > 0$ ($k < 0$, $\Omega < 1$, open Universe), then the system is unbound and it will continue to expand forever, asymptoting to a constant $\dot{a}$, and becoming decreasingly dense and cool and ultimately winding up in a Big Chill. If $E < 0$ ($k > 0$, $\Omega > 1$, closed Universe), then the system is bound; in this case, the Universe will reach a point of maximum expansion and then eventually recollapse to a Big Crunch. If $E = 0$ ($k = 0$, $\Omega = 1$, flat Universe), then the Universe is at the critical density; such a universe will expand forever, asymptotically approaching a zero expansion rate.

So far, everything we have done assumes that the pressures in the Universe are negligible compared with the energy density. This type of pressureless matter is referred to in cosmology as nonrelativistic matter (old cosmology textbooks and papers refer to this as "dust", although this term has a significantly different meaning elsewhere in astronomy)—i.e., objects, such as galaxies, that move with velocities small compared with the speed of light—then $p = 0$. The energy density of such matter simply scales with the scale factor as $\rho \propto a^{-3}$, simply because the number density of particles, and therefore energy density, are inversely proportional to the volume.

If the Universe is filled with relativistic matter, or radiation (i.e., particles that move with $v \simeq c$; e.g., photons or (effectively) massless neutrinos), then $p = \rho/3$. In this case the energy density $\rho \propto a^{-4}$ because the number density decreases as a^{-3} and the energy of each particle decrease as a^{-1} (that is, its wavelength increases as a).

A third possibility which will be relevant is *vacuum energy* matter with an equation of state $p = -\rho$. The energy density of such matter remains constant ($\rho \propto a^0$) as the Universe expands. In general relativity, the addition of vacuum energy is equivalent to the addition of a cosmological constant in Einstein's equation.

More generally, the first law of thermodynamics ($dE = -pdV$) implies $d(\rho a^3) = -pd(a^3)$ or $d[a^3(\rho + p)] = a^3\,dp$. Thus, if the equation of state is $p = w\rho$, with $w =$constant, then $\rho \propto a^{-3(1+w)}$. You can check that this relation holds for nonrelativistic matter ($w = 0$), relativistic matter ($w = 1/3$), and vacuum energy ($w = -1$). In modern cosmology, the parameter w is referred to as the *equation-of-state* parameter, or sometimes, less accurately, as the equation of state. As we will see later, values of $-1 \leq w <\leq 1$ are fairly easy to come by with simple scalar-field models for matter. Values of $w < -1$ or $w > 1$ are much more difficult to come by and violate various energy conditions, but you should be aware that they have been considered in some of the recent literature.

In fact, the Universe consists of several types of matter, and the total stress-energy tensor is the sum of those for the individual components. Throughout much of the history of the Universe, however, the energy density is dominated largely by one component, and so we refer, e.g., to a matter-dominated (MD), radiation-dominated (RD), or vacuum-energy dominated phase.

Let us now return to the second form of the Friedmann equation. In general relativity, it turns out that for spherically symmetric systems, the source for the gravitational field is generalized from ρ to $\rho + 3p$. Thus, the (second) Friedmann equation for the more general equation of state is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3}(\rho + 3p). \tag{20}$$

This form of the equation is particularly interesting, as it shows that there must have been a big bang (i.e., a past point in time where $a \to 0$) even without solving the equation. If $\dot{a} > 0$ today, and if $\rho + 3p > 0$ always and $\Lambda = 0$, then there must have been a point in the past where $a = 0$. Since the energy density for the matter and radiation in the Universe today scale as a^{-3} and a^{-4} today, while that for vacuum energy (cosmological constant) remains constant, then $\rho + 3p$ must have been greater than zero at early times, and so there must have been a big bang.

This equation can also give us the *deceleration parameter*, $q \equiv -a\ddot{a}/\dot{a}^2$. For a Universe with nonrelativistic matter and a cosmological constant, $q_0 = \Omega_m/2 - \Omega_\Lambda$, where the subscript 0 denotes the value today, and Ω_m and Ω_Λ are here their values today. Current measurements suggest $q_0 \simeq -0.55$.

It is useful to list some simple solutions for the Friedmann equation. For a MD Universe, $a \propto t^{2/3}$; for RD, $a \propto t^{1/2}$; and for vacuum-dominated, $a \propto e^{Ht}$. The vacuum-dominated solution is known as a de Sitter spacetime. Such a spacetime has a higher degree of symmetry, there being no preferred time direction as in the other (e.g., MD or RD) FRW Universes. For an empty Universe with no cosmological constant but negative curvature, the Friedmann equation becomes $H^2 = \frac{1}{a^2}$, which has solution $a \propto t$. Such a spacetime, the *Milne spacetime*, has no matter and must therefore be equivalent to a Minkowski. This can be shown with an appropriate change of coordinates.

Finally, it is imperative to point out that our discussion above of the relation between Ω, k, and the ultimate fate of the Universe is valid *only if the Universe consists of nonrelativistic matter.* As we will see, current evidence favors a vacuum-energy (or cosmological-constant) density close to 70% of the critical density, and so our analysis above is not always valid. We thus at this point generalize and clarify our definitions of the Ω parameter(s). We define more generally $\Omega \equiv \rho_{\rm tot}/\rho_c$, where $\rho_c = 3H^2/(8\pi G)$ is the *critical density* and $\rho_{\rm tot}$ is the *total* energy density, including that from a cosmological constant. We can also define similar parameters for the matter component, $\Omega_m = \rho_m/\rho_c$, where ρ_m is the matter density (*not* including cosmological constant), and $\Omega_\Lambda = \rho_\Lambda/\rho_c = \Lambda/3H^2$. Thus, $\Omega = \Omega_m + \Omega_\lambda$ with these definitions. Note that the values of Ω, Ω_m, and Ω_Λ will generally change with time. In most (but not all) cases, the values quoted in the literature refer to their values today. From the Friedmann equation it follows that

$$1 + \frac{k}{H^2 a^2} = \Omega,$$

so the geometry of the Universe is related to the total energy density. The Universe is closed, flat, or open for $\Omega > 1$, $\Omega = 1$, or $\Omega < 1$, respectively.

Expansion History and $\Omega_m - \Omega_\Lambda$ plots.

6 Age of the Universe

To determine the age of the Universe, we simply integrate the time since $t = 0$ (when $a \to 0$; the big bang) until today:

$$t_0 = \int_0^{t_0} dt = \int_0^{a_0} \frac{da}{\dot{a}}. \tag{21}$$

We then recast the Friedmann equation (assuming nonrelativistic matter and a cosmological constant) as an equation for the expansion rate as a function of redshift,

$$H(z) = \frac{\dot{a}}{a} = H_0 E(z), \tag{22}$$

where

$$E(z) = \left[\Omega_m (1+z)^3 + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda)(1+z)^2\right]^{1/2}. \tag{23}$$

Then,

$$t_0 = H_0^{-1} \int_0^\infty \frac{dz}{(1+z)E(z)}. \tag{24}$$

Thus, for example, if we lived in an *Einstein-de Sitter universe* (i.e., $\Omega_m = 1$ and $\Omega_\Lambda = 0$), then $E(z) = (1+z)^{3/2}$ and $t_0 = (2/3)H_0^{-1} \simeq 6.7\, h^{-1}$ Gyr. The integrals for $\Omega_m \neq 1$ and $\Omega_\Lambda \neq 0$ are more complicated but are given for some cases in the usual textbooks. Here, we simply note that the ages are shorter for $\Omega_m > 1$ and $\Omega_\Lambda = 0$ and they are larger if $\Omega_\Lambda = 0$ and $\Omega_m < 1$, or if $\Omega_m = 0$ and $\Omega_m + \Omega_\Lambda = 1$ but $\Omega_\Lambda > 0$. Just as a historical aside, a few years ago, it was believed that globular-cluster ages were as high as 15–20 Gyr, older than the age of the Universe for the then-central values of h, Ω_m, and Ω_Λ. Now, however, the current cosmological parameters indicate an age $t_0 \simeq 13.8$ Gyr, consistent more or less with current globular-cluster ages.

7 Redshift of photons

If a light signal is emitted with wavelength λ_e by a comoving object at some time t_e, then when it is observed by a comoving observer at some later time t_o, it will have a wavelength

$$\lambda_o = \lambda_e \frac{a(t_o)}{a(t_e)} \equiv 1 + z.$$

This also defines the *redshift* z. This result follows heuristically by noting that the wavelength of the photon increases with the scale factor of the Universe, but it can be derived formally (as you will do in a homework problem) from the geodesic equation or by identifying conserved quantities associated with Killing vectors of the FRW spacetime. This formula agrees for $z \ll 1$ with our earlier result, where we associated the redshift with the Doppler shift from a recessional velocity, but it also applies to the case where $z \gtrsim 1$. In cosmology, when we talk about a galaxy "at a redshift z", we are referring to a galaxy that we see now when the size of the Universe was a factor $(1+z)^{-1}$ smaller than it is now. The most distant galaxies observed so far have $z \sim 7$. As we will see later, the cosmic microwave background was emitted from $z \simeq 1100$.

With the geodesic equation, it can also be shown that a massive particle emitted by a comoving observer at time t_e with momentum p_e will be seen to have a momentum

$$p_o = p_e \frac{a(t_e)}{a(t_o)} = \frac{1}{1+z},$$

when it passes a comoving observer at time t_o. This follows heuristically by noting that the de Broglie wavelength $\lambda \propto p^{-1}$ increases with $a(t)$.

8 Horizons

Recall the FRW metric in the form,

$$ds^2 = dt^2 - a^2(t) \left[d\chi^2 + \begin{pmatrix} \sin^2 \chi \\ \chi^2 \\ \sinh^2 \chi \end{pmatrix} (d\theta^2 + \sin^2 \theta \, d\phi^2) \right], \tag{25}$$

and also recall that photons move along geodesics, $ds = 0$. Consider a photon emitted at time t_e from the origin ($\chi = 0$) that moves in the direction $\theta = \phi = 0$. Then, from $ds = 0$, it follows that the photon moves along a trajectory with $dt = a d\chi$, and therefore

$$\chi = \int_{t_e}^{t_0} \frac{dt}{a(t)}, \tag{26}$$

is the coordinate distance traveled by the photon between times t_e and t_o. For example, in a flat MD Universe, $a \propto t^{2/3}$ and so the physical distance traveled by the photon is

$$a_0 \chi = 3t_0[1 - (t_e/t_0)^{1/3}] = 3t_0[1 - (1 + z_e)^{-1/2}]. \tag{27}$$

In such a model, the maximum distance traveled by a photon between time $t = 0$ and today is $a_0 r = 3t_0 = 2/H_0 = 6000\, h^{-1}$ Mpc, where we have written the Hubble constant as $H_0 = 100\, h$ km/sec/Mpc, and, as mentioned above, the numerical value is $h \sim 0.7$.

For our Universe, which has $\Omega_m \simeq 0.3$ and $\Omega_\Lambda \simeq 0.7$, the numerical value for the horizon distance is slightly different. This means that there is a finite observable volume for the Universe.