Alternative-gravity theories

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1 Field theory

In this last week of class I'm going to jump ahead and presume that you're familiar with classical field theory and also with general relativity. The subject for this week will be alternative-gravity theories, and in this week we'll get a lightning introduction to a very active area of research. The original motivation for the current generation of investigations of alternative-gravity theories was the discovery of cosmic acceleration. While the simplest explanation is an addition of a cosmological constant to general relativity, the ridiculoulsy small value (in the natural dimensionless units) of the cosmological constant leads theorists to speculate that perhaps something else is going on. Beyond that, though, theorists have a continuing interest in studying the structure of theories, gravity theories included among them.

Let's begin by recalling that a *scalar field* $\Phi(x^\mu)$ is a Lagrangian density $\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi)$, which gives the Lagrangian $L = \int d^3x \mathcal{L}$ and action $S = \int dt\, L = \int d^4x \mathcal{L}$. The equation of motion for the scalar field is obtained by varying the action to obtain the equation of motion (the Euler-Lagrange equation),

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) = 0, \tag{1}$$

for the scalar field.

If the scalar field has kinetic-energy density $(1/2)\dot{\phi}^2$, gradient-energy density $(1/2)(\nabla\phi)^2$, and a potential-energy density $V(\phi)$, then the Lagrangian density $\mathcal{L} = (1/2)\dot{\phi}^2 - (1/2)(\nabla\phi)^2 - V(\phi)$, which we can write in the Lorentz-covariant form,

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi). \tag{2}$$

Using $\partial\mathcal{L}/\partial\Phi = -dV/d\Phi$ and $\partial\mathcal{L}/\partial(\partial_\mu\Phi) = -\eta^{\mu\nu}\partial^\nu\Phi$, we obtain the equation of motion,

$$\Box\Phi = dV/d\phi, \tag{3}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the *d'Alembertian.* If $V(\Phi) = 0$, then we get $\Box\Phi = 0$, the wave equation, which has propagating wavelike solutions that travel at the speed of light. We infer from this, in analogy with electromagnetism, that when quantized, the quanta of this field will be massless scalar particles; in this case, we call Φ a "massless scalar field." If $V(\phi) = (1/2)m^2\Phi^2$, then the equation of motion is $\Box\Phi - m^2\Phi = 0$, which, when Fourier transformed from $t, \vec{x}$ to $E, \vec{p}$ leads to a relation $E^2 = p^2 + m^2$ between the energy E and momentum p. From this we infer that the quanta of the scalar field are in this case scalar particles of mass m.

So, let's now move on to general relativity. Before going further, it is fair to ask why we even bother with a Lagrangian formulation of general relativity. Everything in GR follows from the equations of motion for the gravitational field $g_{\mu\nu}$. So why do we need a Lagrangian from which to derive the Einstein equation? Strictly speaking, if we're sticking to classical GR, there's no immediate advantage. But if we want to begin to try to understand how, for example, GR emerges in some higher "meta-"theory (e.g., string theory), then it may be the Lagrangian of GR—rather than its equation of motion—that appears in such theories. Likewise, it may often be easier or more common to try to formulate possibilities for alternatives or extensions to GR with the Lagrangian than with the equation of motion. Finally, there may actually be some situations where it is easier to determine the spacetime by directly varying the action, rather than by trying to identify the relevant components of the Einstein equation.

In GR, the fields are the metric components $g_{\mu\nu}$. We want to find an action, some functional of $g_{\mu\nu}$ that, when varied with respect to $g_{\mu\nu}$, gives the Einstein equation. We know that the field equations are quadratic in derivatives of $g_{\mu\nu}$. We can therefore guess that the Lagrangian should be quadratic in derivatives of $g_{\mu\nu}$. It must also be a scalar. The simplest guess for the action is thus

$$S_H = \text{constant} \int \sqrt{-g}\, d^4x\, R. \tag{4}$$

It turns out that this guess, known as the *Einstein-Hilbert action*, is the correct guess, as shown in Carroll's book. Extremizing the action gives $G_{\mu\nu} = 0$. To include matter, we need to add an action S_m for matter to the Einstein-Hilbert action. We thus write the total action as

$$S = \frac{1}{16\pi G} S_H + S_M, \tag{5}$$

where the factor of $16\pi G$ dividing S_H so we recover $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ correct Einstein equation if we identify the energy-momentum tensor to be

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{6}$$

The Lagrangian approach allows us to study alternative (to GR) theories of gravity. Perhaps the most widely studied of such theories are "scalar-tensor" theories, and perhaps the most widely studied of these are "Brans-Dicke" theories. One reason scalar-tensor theories are of interest is that they show up in string theory, although the study of scalar-tensor theories precedes the advent of string theory by a considerable amount. Scalar-tensor theories introduce a scalar field $\lambda(x^\mu)$ and are defined by the action $S = S_{fR} + S_\lambda + S_M$, where

$$S_{fR} = \int d^4x\, \sqrt{-g} f(\lambda) R, \tag{7}$$

$$S_\lambda = \int d^4x\, \sqrt{-g}\left[-\frac{1}{2}h(\lambda) g^{\mu\nu}(\partial_\mu\lambda)(\partial_\nu\lambda) - U(\lambda)\right], \tag{8}$$

and S_M is the usual matter Lagrangian. A particular scalar-tensor theory is thus given by specifying the functions $f(\lambda)$, $h(\lambda)$, and $U(\lambda)$. Variation with respect to the inverse metric gives

$$\delta S_{fR} = \int d^4x \sqrt{-g} f(\lambda) \left[\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} + \nabla_\sigma \nabla^\sigma (g_{\mu\nu} \delta g^{\mu\nu}) - \nabla_\mu \nabla_\nu (\delta g^{\mu\nu}) \right]. \tag{9}$$

When dealing with Einstein action, we argued that the last two terms vanished because they were total derivatives that could be converted to surface terms. However, there is a function $f(\lambda)$ which is implicitly a function of spacetime coordinates multiplying the derivative, and so the last two terms can no longer be converted to surface terms. We thus integrate by parts twice to put those derivatives on f:

$$\delta S_{fR} = \int d^4x \sqrt{-g} \left[f(\lambda) G_{\mu\nu} + g_{\mu\nu} \Box f - \nabla_\mu \nabla_\nu f \right] \delta g^{\mu\nu}. \tag{10}$$

This gives us the scalar-tensor field equation,

$$G_{\mu\nu} = f^{-1}(\lambda) \left(\frac{1}{2} T^{(\mathrm{M})}_{\mu\nu} + \frac{1}{2} T^{(\lambda)}_{\mu\nu} + \nabla_\mu \nabla_\nu f - g_{\mu\nu} \Box f \right), \tag{11}$$

for the metric, where $T^{(\mathrm{M})}_{\mu\nu}$ is the matter energy-momentum tensor and

$$T^{(\lambda)}_{\mu\nu} = h(\lambda)(\nabla_\mu \lambda)(\nabla_\nu \lambda) - g_{\mu\nu} \left[\frac{1}{2} h(\lambda) g^{\rho\sigma} \nabla_\rho \lambda \nabla_\sigma \lambda + U(\lambda) \right]. \tag{12}$$

Note that if the scalar field is constant in spacetime, then general relativity is recovered by identifying $f(\lambda) = 1/16\pi G$; thus scalar-tensor theories are often viewed as theories with variable Newton's constant. There is also an equation of motion for the scalar field, and that is obtained by varying the action with respect to λ. The equation of motion is

$$h \Box \lambda + \frac{1}{2} h' g^{\mu\nu} (\nabla_\mu \lambda)(\nabla_\nu \lambda) - U' + f' R = 0. \tag{13}$$

Brans-Dicke theory is a special case of scalar-tensor theory, parameterized by a quantity ω. The theory is specified by

$$f(\lambda) = \frac{\lambda}{16\pi}, \qquad h(\lambda) = \frac{\omega}{8\pi} \qquad U(\lambda) = 0. \tag{14}$$

The scalar field in this theory is massless and thus gives rise to a long-range propagating scalar degree of freedom (in addition to the two tensorial degrees of freedom in ordinary GR). There are thus scalar gravitational waves in this theory (in addition to the two tensorial modes), and spherically-symmetric pulsations of a star and give rise to such gravitational waves. In the limit that $\omega \to \infty$, the energy cost for scalar-field variations in space or time become huge; the scalar field in this limit thus becomes nondynamical (i.e., frozen to a single value), and GR is recovered. One consequence of Brans-Dicke theory, which you will work out in a homework assignment, is to change GR's prediction for the deflection of light by the Sun. Current measurements find light deflection in extremely good agreement with the predictions of GR, and this constrains $\omega \gtrsim 3000$.

The action, as we've studied above, is written in the *Jordan frame* or *string frame*. In this frame, it is the metric $g_{\mu\nu}$ that determines, through the geodesic equation, the trajectories of freely-falling particles. An alternative way to look at these theories is to make a transformation to a theory with a new metric,

$$\tilde{g}_{\mu\nu} = 16\pi \tilde{G} f(\lambda) g_{\mu\nu}, \tag{15}$$

where $\tilde{G}$ is a constant that will resemble (but not necessarily be equal to) a Newton constant. With this transformation, the fR action becomes (after integrating by parts and discarding a surface term),

$$S_{fR} = \int d^4x \sqrt{-\tilde{g}}(16\pi\tilde{G})^{-1} \left[\tilde{R} - \frac{3}{2}\tilde{g}^{\rho\sigma} f^{-2} \left(\frac{df}{d\lambda}\right)^2 (\tilde{\nabla}_\rho\lambda)(\tilde{\nabla}_\sigma\lambda)\right]. \quad (16)$$

With this transformation, the gravitational action winds up looking like ordinary GR with the addition of a kinetic term for the scalar field. Keep in mind, though, that freely-falling particles follow geodesics of $g_{\mu\nu}$, *not* $\tilde{g}_{\mu\nu}$, and so the theory is quite different than GR with a scalar field, even though the action looks similar. The action of the theory written in terms of $\tilde{g}_{\mu\nu}$ is called the *Einstein frame*. One interesting consequence of this transformation is that even if the original theory has $h(\lambda) = 0$ (and thus no kinetic term for the scalar field), there is still a kinetic term in the Einstein frame. Thus, the scalar degree of freedom is propagating in scalar-tensor theory, even if there is no explicit kinetic term in the Jordan frame.

Carroll shows that by defining a new field by $\phi = \int K^{1/2}\, d\lambda$, where

$$K(\lambda) = \frac{1}{16\pi\tilde{G}f^2}\left[fh + 3(f')^2\right], \quad (17)$$

the gravitational action can be written

$$S_{fR} + S_\lambda = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi\tilde{G}} - \frac{1}{2}\tilde{g}^{\rho\sigma}(\tilde{\nabla}_\rho\phi)(\tilde{\nabla}_\sigma\phi) - V(\phi)\right], \quad (18)$$

with

$$V(\phi) = \frac{U(\lambda(\phi))}{(16\pi\tilde{G})^2 f^2(\lambda(\phi))}. \quad (19)$$

The energy-momentum tensor in the Einstein frame is obtained by varying S with respect to $\tilde{g}_{\mu\nu}$. The result is $\tilde{T}_{\mu\nu} = (16\pi\tilde{G}f)^{-1}T_{\mu\nu}$. Also, since $g^{\alpha\beta} = 16\pi\tilde{G}f\tilde{g}^{\alpha\beta}$, there is an explicit dependence on the scalar field in S_M. Thus, when the complete action is varied with respect to ϕ (to obtain the equation of motion for ϕ), it gives a matter source for ϕ:

$$\tilde{\Box}\phi - \frac{dV}{d\phi} = \frac{1}{2f}\frac{df}{d\phi}\tilde{T}^{(\mathrm{M})}, \quad (20)$$

where $\tilde{T}^{(\mathrm{M})} = \tilde{g}^{\alpha\beta}\tilde{T}^{(\mathrm{M})}_{\alpha\beta}$. The Einstein equation in the Einstein frame is then just $\tilde{G}_{\mu\nu} = 8\pi\tilde{G}\left(\tilde{T}^{(\mathrm{M})}_{\mu\nu} + T^{(\phi)}_{\mu\nu}\right)$, with

$$\tilde{T}^{(\phi)}_{\mu\nu} = \tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\phi - \tilde{g}_{\mu\nu}V(\Phi). \quad (21)$$

In the Einstein frame, the physics looks like ordinary GR with matter fields and a scalar field. The difference, though, is that the metric $\tilde{g}_{\mu\nu}$ is *not* the metric whose geodesics determine the trajectories of unaccelerated particles. It is the original metric $g_{\mu\nu}$ in the Jordan frame that determines these trajectories. So there is indeed a difference.

Extra dimensions: Another class of alternative-gravity theories can be developed by allowing for the possibility that the Universe contains extra spatial dimensions, in addition to the three we know about. But how can this happen if we only "see" three? One possibility is that an extra dimension(s) can be "curled up" into a microscopic distance. For example, suppose I start with a sheet of paper, a two-dimensional manifold. Suppose I then roll it up into a cylinder, and then I wind the cylinder very tightly, so that the radius R of the cylinder is very small $R \ll l$ compared with the length l of the cylinder. In this case, the two-dimensional surface, the cylinder, looks effectively like a one-dimensional manifold, a string. Likewise, there could be spatial dimensions, in addition to the three we know about, that are compact and too small to be noticed. As we will explore in the subsequent discussion and homework problems, such models could be interesting for a number of reasons. Extra dimension arise in string theory, and they have been very popular in particle theory and cosmology recently.

Let G_{ab} be the metric for the $(4+d)$-dimensional spacetime with coordinates X^a, where a runs from 0 to $d+3$. Let's consider a simple example, in which the line element is

$$ds^2 = G_{ab} dX^a dX^b = g_{\mu\nu}(x) dx^\mu dx^\nu + b^2(x)\gamma_{ij} dy^i dy^j, \tag{22}$$

where x^μ are the ordinary (3+1)-d coordinates, and the y^i are coordinates on the extra-dimensional manifold, and $\gamma_{ij}(y)$ is the metric on that manifold. The complete action is

$$S = \int d^{4+d}X\sqrt{-G}\left(\frac{1}{16\pi G_{4+d}} R[G_{ab}] + \tilde{\mathcal{L}}_M\right). \tag{23}$$

Since the scale factor $b(x)$ in this example does not depend on the extra coordinates y, we can integrate over the extra dimensions. We use $\sqrt{-G} = b^d\sqrt{-g}\sqrt{\gamma}$, and we can evaluate the Ricci scalar,

$$\begin{aligned} R[G_{ab}] = \quad & R[g_{\mu\nu}] + b^{-2}R[\gamma_{ij}] - 2db^{-1}g^{\mu\sigma}\nabla_\mu\nabla_\sigma b \\ & -d(d-1)b^{-2}g^{\mu\sigma}(\nabla_\mu b)(\nabla_\sigma b), \end{aligned} \tag{24}$$

where ∇_μ is the derivative associated with $g_{\mu\nu}$. The volume of the extra dimension, when $b=1$, is $\mathcal{V} = \int d^d y\sqrt{\gamma}$. We can tell from dimensional arguments that the extra-dimensional Newton's constant G_{4+d} is different than the ordinary Newton's constant in our usual $(3+1)$-d world. The two are related by

$$\frac{1}{16\pi G_4} = \frac{\mathcal{V}}{16\pi G_{4+d}}; \tag{25}$$

i.e., the usual Newton constant is the extra-dimensional one divided by the volume of the extra dimensions. The action can then be written,

$$\begin{aligned} S = \quad & \int d^4x\sqrt{-g}\left\{\frac{1}{16\pi G_4}\left[b^d R[g_{\mu\nu}] + d(d-1)b^{d-2}g^{\mu\nu}(\nabla_\mu b)\right.\right. \\ & \left.\left.(\nabla_\nu b) + d(d-1)\kappa b^{d-2}\right] + \mathcal{V}b^d\mathcal{L}_M\right\}, \end{aligned} \tag{26}$$

where there has been an integration by parts and $\kappa = R[\gamma_{ij}]/[d(d-1)]$ is a parameter that describes the curvature of the extra dimensions. If you take a good look at this action, you'll see that it is precisely the same as the action for a scalar-tensor theory, with the size $b(x)$ of the extra dimensions playing the role of the scalar field. This is, more or less, the origin of the often-heard statement that string theory predicts that gravity is a scalar-tensor theory. In string theory, there are something like 5 or 6 extra spatial dimensions that get "compactified" relative to the large three that we see, and this compactification can lead to an effective $(3+1)$-d scalar-tensor theory of gravity. One can then go a bit further (see Carroll's book) and re-write this theory in the Einstein frame, rather than Jordan frame. The canonically-normalized scalar field that results is then referred to as the "dilaton" or "radion" field.

f(R) theories: So we've now seen that the scalar-tensor theory can also serve as a proxy for an extra-dimensional theory. We'll now see that another class of theories, $f(R)$ theories, are also equivalent to scalar-tensor theories. In $f(R)$ gravity theories, the curvature R in the Einstein-Hilbert action is replaced by a function $f(R)$:

$$\frac{1}{16\pi G}\int d^4x\sqrt{-g}[R+\mathcal{L}_M]. \tag{27}$$

Variation of this action then leads to the generalization of the Einstein equation,

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} - \nabla_\mu\nabla_\nu f_R + \Box f_R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{28}$$

where $f_R \equiv df/dR$. As an example, one particular form that has received a lot of attention is $1/R$ gravity, where $f(R) = R - \mu^4/R^2$. The field equation in this theory is

$$\left(1+\frac{\mu^4}{R^2}\right)R_{\mu\nu} - \frac{1}{2}\left(1-\frac{\mu^4}{R^2}\right)Rg_{\mu\nu} + \mu^4(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu)R^{-2}. \tag{29}$$

The motivation for studying this theory is that it may provide a solution to the cosmic-acceleration puzzle. As we have seen, one way to explain cosmic acceleration is the addition of a cosmological constant to Einstein's equations, or equivalently, the addition of a fluid with equation of state $p=-\rho$. With such an addition, the vacuum solution to Einstein's equations is the de Sitter spacetime, an FRW Universe with scale factor $a(t)\propto e^{Ht}$. This $1/R$ gravity provides another way to get a de Sitter spacetime as a vacuum solution. To see this, take the trace (i.e., contract with $g^{\mu\nu}$) of the field equation to obtain

$$\Box\frac{\mu^4}{R^2} - \frac{R}{3} + \frac{\mu^4}{R} = \frac{8\pi G}{3}T, \tag{30}$$

where $T\equiv g^{\mu\nu}T_{\mu\nu}$. The constant-curvature ($\nabla_\mu R = 0$) vacuum ($T=0$) solution is thus $R=\sqrt{3}\mu^2$; i.e., a de Sitter spacetime with Hubble parameter $H^2 = \mu^2/(4\sqrt{3})$.

We will now show that $f(R)$ theory is equivalent to a scalar-tensor theory. To do so, we introduce a scalar field $\phi(x)$, which acts as a Lagrange multiplier, and re-write the action as

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,\left[f(\phi)+f'(\phi)(R-\phi)+\mathcal{L}_M\right], \tag{31}$$

where $f'(\phi) = df/d\phi$. Minimizing the action with respect to ϕ gives a field equation $\phi = R$ [if $f''(\phi)\neq 0$], and then plugging $\phi = R$ back into the action gives us the original $f(R)$-theory action. We can then use all the machinery for scalar-tensor theories to study any given $f(R)$ theory.

Solar system tests: General relativity makes a very specific prediction for the deflection of light by the Sun, a prediction in very precise agreement with measurements. Most generally, an alternative theory will make a different prediction. Here we will show, as an example, that the $1/R$

gravity theory, introduced to explain cosmic acceleration, disagrees with Solar System tests. In a homework, you will generalize this result to Brans-Dicke theories.

The action,

$$S = \int d^4x\,\sqrt{-g}\left(R + \frac{\mu^4}{R} + \mathcal{L}_{\rm matter}\right), \tag{32}$$

of $1/R$ gravity can be mapped onto a scalar-tensor theory with action,

$$S = \int d^4x\sqrt{-g}\left\{f(\phi)R - U(\phi) + \mathcal{L}_{\rm matter}\right\}, \tag{33}$$

by identifying

$$f(\phi) = \frac{1}{16\pi G}\left(1 + \frac{\mu^4}{\phi^2}\right), \qquad U(\phi) = \frac{1}{8\pi G}\frac{\mu^4}{\phi}. \tag{34}$$

Now change variables by redefining $\varphi = 1 + (\mu^4/\phi)$, so that $f(\varphi) = \varphi/(16\pi G)$, and $U(\varphi) = (\mu^2/8\pi G)\sqrt{\varphi - 1}$. The condition $U' = f'R$ (obtained by varying the action with respect to φ; φ in these theories is thus a Lagrange multiplier) gives us the constraint,

$$R = \frac{\mu^2}{\sqrt{\varphi - 1}} = \phi. \tag{35}$$

The equation of motion (the Einstein equation) for the metric is now [see, e.g., Section 4.8 in Carroll],

$$G_{\mu\nu} = \frac{8\pi G}{\varphi}T_{\mu\nu} - \frac{8\pi G}{\varphi}U(\varphi)g_{\mu\nu} + \frac{1}{\varphi}\nabla_\mu\nabla_\nu\varphi - \frac{1}{\varphi}g_{\mu\nu}\Box\varphi. \tag{36}$$

Contracting with the inverse metric $g^{\mu\nu}$ on both sides (using $g^{\mu\nu}g_{\mu\nu} = 4$ and $g^{\mu\nu}G_{\mu\nu} = -R$),

$$-R = -\frac{\mu^2}{\sqrt{\varphi - 1}} = \frac{8\pi G}{\varphi}T - \frac{32\pi G}{\varphi}U(\varphi) - \frac{3}{\varphi}\Box\varphi, \tag{37}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. The vacuum solution is obtained by setting $T = 0$ and $\nabla_\mu R = \nabla_\mu\phi = 0$; it is

$$-\frac{32\pi G}{\varphi}\frac{\mu^2}{8\pi G}\sqrt{\varphi - 1} = -\frac{\mu^2}{\sqrt{\varphi - 1}}, \tag{38}$$

or, with a little algebraic rearrangement, $\varphi = 4/3$. Equivalently,

$$\phi = \frac{3}{\mu}^2 = R, \tag{39}$$

which we recognize as the scalar curvature for the de-Sitter spacetime, the spacetime of constant positive curvature. We write the metric for this spacetime as

$$ds^2 = -(1 - H^2r^2)dt^2 + (1 - H^2r^2)^{-1}dr^2 + r^2d\Omega^2, \tag{40}$$

where $H^2 = R/12$; i.e., as a spherically-symmetric static spacetime.

We now consider the spacetime in the Solar System in this theory. First of all, the distance $\sim 10^{13}$ cm in the Solar System are tiny compared with the distance $H^{-1} \sim 10^{27}$ cm, so $rH \ll 1$ everywhere in the Solar System. Moreover, the densities and velocities in the Solar System are sufficiently small that we can treat the spacetime as a small perturbation to the de Sitter spacetime. The spacetime should also be spherically symmetric and static. The most general static spherically symmetric perturbation to the de Sitter spacetime can be written

$$ds^2 = -[1 - H^2r^2 + a(r)]dt^2 + [1 + H^2r^2 + b(r)]dr^2 + r^2 d\Omega^2, \tag{41}$$

where the metric-perturbation variables $a(r), b(r) \ll 1$. In the following, we work to linear order in a and b, and also recall that $rH \ll 1$. However, a, b are *not* necessarily small compared with rH.

We now return to Eq. (37),

$$\Box\varphi = \frac{8\pi G}{3}T + \frac{\mu^2}{3}\left(\frac{3\varphi - 4}{\sqrt{\varphi - 1}}\right), \tag{42}$$

noting that this is an *exact* equation for φ. We then write small perturbations to the unperturbed solution $\varphi = 4/3$ as

$$\varphi = \frac{4}{3} + c(r), \tag{43}$$

with $c(r) \ll 1$. Then, the equation of motion for φ becomes

$$\Box c = \frac{8\pi G}{3}T + \frac{\sqrt{3}\mu^2}{2}c, \tag{44}$$

as long as $c \ll 1$. In the Newtonian limit appropriate for the Solar System, the pressure p is negligible with the energy density ρ, and so $T = \rho$, and

$$\nabla^2 c - \frac{\sqrt{3}}{2}\mu^2 c = \frac{8\pi G}{3}\rho. \tag{45}$$

The Green's function for the left-hand side of this equation is $e^{-mr}/(4\pi r)$, with $m = 3^{1/4}\mu/\sqrt{2} = \mathcal{O}(H)$, so in the Solar System, we may use the Green's function $(1/4\pi r)[1 + \mathcal{O}(rH)]$. Equivalently, the equation for c reduces to $\nabla^2 c = 8\pi G\rho/3$. Integrating the right-hand side over a spherical volume of radius r gives us $8\pi G m(r)/3$, where $m(r)$ is the mass enclosed by a radius r. Using Stokes' law to integrate the right-hand side gives us $-4\pi r^2(dc/dr)$. Thus, the equation for $c(r)$ becomes

$$\frac{dc}{dr} = -\frac{2Gm(r)}{3r^2}\left[1 + \mathcal{O}(rH)\right]. \tag{46}$$

Integrating this equation from $r = \infty$ [where $c(\infty) = 0$ so that the solution matches onto the unperturbed solution] to some radius $r > R_\odot$ gives us the solution $c(r) = (2/3)(GM/r)[1+\mathcal{O}(rH)]$ for $r > R_\odot$. Note that this result remains consistent with our assumption $c(r) \ll 1$ and will remain consistent even for $r < R_\odot$ for a density $\rho(r) \rightarrow$constant as $r \rightarrow 0$. Thus,

$$\varphi(r) = \frac{4}{3} + \frac{2}{3}\frac{GM}{r}, \qquad r > R_\odot, \tag{47}$$

implying (using $\varphi = 1 + \mu^4/R^2$) that

$$R = \sqrt{3}\mu^2\left(1 - \frac{GM}{r}\right), \qquad r > R_\odot. \tag{48}$$

We have thus shown that $R \neq$constant outside the star and have thus already arrived at a result at odds with the constant-curvature de Sitter-Schwarzchild solution. Notice that had we (*incorrectly!*) used $\rho = 0$ in Eq. (45), then the equations would have admitted the solution $c(r) = 0$; i.e., the constant-curvature solution. However, *this would be incorrect, because even though* $\rho = 0$ *at* $r > R_\odot$, *the solution to the differential equation at* $r > R_\odot$ *depends on the mass distribution* $\rho(r)$ *at* $r < R_\odot$*!* In other words, although the de Sitter-Schwarzchild solution is a static spherically-symmetric solution to the vacuum Einstein equations, *it is not the solution that correctly maps onto the solution inside the star.* Note also that although we have derived this result using the language of scalar-tensor theory, Eq. (37) is identical to the trace of the Einstein equation for $1/R$ theory. Note also that integration of the equation for $c(r)$ to radii $r < R_\odot$ inside the star, implies that the scalar curvature R remains of order μ^2, even inside the star. Thus the theory produces a spacetime inside the star that is *very* different from general relativity, which predicts a curvature $R = 8\pi G\rho/3$ inside the star.

To proceed to the solutions for $a(r)$ and $b(r)$, we trace-reverse the Einstein equation,

$$R_{\mu\nu} = \frac{8\pi G}{\varphi}(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) + \frac{8\pi G}{\varphi}U(\varphi)g_{\mu\nu} \tag{49}$$

$$+\frac{1}{\varphi}\left[\nabla_\mu\nabla_\nu + \frac{1}{2}(\Box\varphi)g_{\mu\nu}\right]. \tag{50}$$

With our solution for $\varphi(r)$, we now have

$$\frac{8\pi G}{\varphi}U(\varphi) = \frac{\sqrt{3}}{4}\mu^2\left[1 + \mathcal{O}\left(\frac{GM}{r}\right)\right]. \tag{51}$$

For the metric, Eq. (41), the tt component of the Ricci tensor is (to linear order in small quantities) $R_{tt} - 3H^2 + (1/2)\nabla^2 a(r)$, so the tt component of the trace-reversed Einstein equation becomes (after the zero-th order terms cancel,

$$\frac{1}{2}\nabla^2 a = -4\pi G\rho, \tag{52}$$

plus terms that are higher order in GM/r and rH. The solution to this equation parallels that for $c(r)$; it is

$$a(r) = -\frac{2GM}{r}, \qquad r > R_\odot, \tag{53}$$

which recovers the Newtonian limit for the motion of nonrelativistic bodies in the Solar System, as it should.

The rr component of the Ricci tensor is (to linear order in small quantities) $R_{rr} = 3H^2 + (b'/r) - (a''/2)$, where the prime denotes derivative with respect to r. The rr component of the trace-reversed Einstein equation is (after the zeroth-order terms cancel),

$$\frac{b'}{r} = \frac{a''}{2} - 3\pi G\rho + \frac{3}{4}c'' + \frac{1}{2}\frac{3}{4}\nabla^2\varphi, \tag{54}$$

which may be re-written (using $\nabla^2\varphi = 8\pi G\rho/3$),

$$\frac{1}{r}\frac{db}{dr} = \frac{1}{2}a'' + \frac{3}{4}c'' - 2\pi G\rho. \tag{55}$$

Plugging in the solutions for $a(r)$ and $c(r)$ and integrating subject to the boundary condition $b(r) \to 0$ as $r \to \infty$, we find,

$$b(r) = \frac{GM}{r}, \qquad r > R_\odot. \tag{56}$$

The metric outside the star thus becomes,

$$ds^2 = -\left(1 - H^2r^2 - \frac{2GM}{r}\right)dt^2 \tag{57}$$

$$+\left(1 + H^2r^2 + \frac{GM}{r}\right)dr^2 + r^2 d\Omega^2. \tag{58}$$

In the study of alternative-gravity theories, there is a parameterization of the possible Solar System spacetimes written in terms of several *parameterized post-Newtonian (PPN) parameters.* The simplest of these PPN parameters, γ, is defined by the metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{2\gamma GM}{r}\right)dr^2 + r^2 d\Omega^2, \tag{59}$$

outside the Sun. Noting that in the Solar System, $rH \ll 1$, we find that the PPN parameter $\gamma = 1/2$ for $1/R$ theory.

Massive gravity: In GR, gravity is mediated by a massless spin-2 field. But what if the graviton had a tiny mass m? Heuristically, we'd expect in this case the $1/r$ potential of gravity to weaken at $r \gtrsim m^{-1}$ to e^{-mr}/r, and one might wonder whether some such modification of GR with $m \sim H^{-1}$ could have something to do with cosmic acceleration. Adding a small mass to GR, though, is not as simple as you'd think, the basic reason being that if the graviton has a mass, then there are 5, rather than 2, dynamical degrees of freedom, and these do not necessarily vanish or decouple in the $m \to 0$ limit. Thus, massive gravity does not constitute a small perturbation to GR, even in the $m \to 0$ limit. The development of ideas for massive-gravity theories that recover GR in the $m \to 0$ limit has been a big deal in theoretical physics recently. Here we'll only scratch the surface of these developments. (My discussion here is taken from Damour, Kogan, and Papazoglou, *Phys. Rev. D* bf 67, 064009 (2003)).

We'll restrict our attention to linearized gravity and write the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu} \ll 1$. Einstein's equations are then

$$\Box h_{\mu\nu} + h^{\lambda}_{\mu,\lambda\nu} - \eta_{\mu\nu}h^{\kappa\lambda}_{,\kappa\lambda} - h_{,\mu\nu} + \eta_{mu\nu}\Box h = 16\pi G T_{\mu\nu}, \tag{60}$$

where $h = h^{\mu}_{\mu}$. In the proper gauge, and for $T_{\mu\nu} = 0$, this reduces to the wave equation $\Box h_{\mu\nu} = 0$. In order to introduce a mass, we'd like to modify this equation so that in the appropriate limit it turns the wave equation into a Klein-Gordon equation $(\Box + m^2)h_{\mu\nu} = 0$. To do so, before choosing a gauge, we make an *ansatz* and add to the left-hand side (which contains only terms with two derivatives) all terms $(0, 2)$ tensors linear in h: i.e., $m^2(h_{\mu\nu} - \alpha\eta_{\mu\nu}h)$; i.e.,

$$\begin{aligned}\Box h_{\mu\nu} \;\; + \;\; & h^{\lambda}_{\mu,\lambda\nu} - \eta_{\mu\nu}h^{\kappa\lambda}_{,\kappa\lambda} - h_{,\mu\nu} + \eta_{mu\nu}\Box h \\ & + m^2(h_{\mu\nu} - \alpha\eta_{\mu\nu}h) = 16\pi G T_{\mu\nu},\end{aligned} \tag{61}$$

where α is a constant that will shown to be 1.

Since $\partial^{\mu}T_{\mu\nu} = 0$, we must have $h^{;\nu}_{\mu\nu} = \alpha h_{,\mu}$. Then taking the trace of Eq. (61), we obtain,

$$2(1-\alpha)\Box h + m^2(1-4\alpha)h = 16\pi G T. \tag{62}$$

Suppose we choose $\alpha = 1$. If so, then, this becomes an algebraic constraint equation for the trace h. This then leaves us with 5 (rather than six—10 components of $h_{\mu\nu}$ minus the four gauge degrees of freedom) dynamical degrees of freedom in $h_{\mu\nu}$, which is what we want for a massive spin-2 field. If $\alpha \neq 1$, then when one writes the Lagrangian that gives rise to Eq. (61), the kinetic-energy term associated with this sixth degree of freedom not only exists (which is a problem), but also has a negative sign (i.e., its a “ghost” degree of freedom), which is additionally problematic. It is thus generally (although not universally) presumed that the mass term in Eq. (61) must be of the “Pauli-Fierz” form, with $\alpha = 1$.

If we now take the $\alpha = 1$, then the linearized Einstein equation, which in ordinary GR takes the form,

$$\Box h_{\mu\nu} = 16\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), \tag{63}$$

then becomes

$$(\Box + m^2) h_{\mu\nu} = 16\pi G \left(T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu} \right) + \frac{1}{3m^2} T_{,\mu\nu}. \tag{64}$$

As you will notice, this equation does not recover GR as $m \to 0$. Even ignoring the last term, the 1/2 in the GR equation is replaced by a 1/3, which you can show is what arises in Brans-Dicke theory with $\omega = 0$. The fact that GR is not recovered as $m \to 0$ is a consequence of the fact that when $m \neq 0$, we have a system with 5 dynamical degrees, as opposed to two, and those three additional degrees of freedom do not simply disappear if we let $m \to 0$.

Confused? Don't feel alone; so is everyone else. Actually, there is an active theoretical effort to figure out how to construct a self-consistent massive-gravity theory that recovers GR smoothly as $m \to 0$. Although there is still no clear and simple story, there have been many interesting developments recently. The interest in these theories has been enhanced by the realization that many extra-dimensional models (e.g., DGP gravity), in which the extra dimensions are implemented differently than we implemented them above, look like massive-gravity theories. For a nice recent review, see arXiv:1105.3735 (“Theoretical Aspects of Massive Gravity,” by Kurt Hinterbichler) and for alternative-gravity theories more generally, see arXiv:1106.2476 (“Modified Gravity and Cosmology,” by Clifton et al.).