Week 4: The growth of density perturbations

January 9, 2012

1 Brief Overview

Over the past few decades, a number of surveys (most recently the Sloan Digital Sky Survey and the Two-Degree Field) have measured the redshifts of thousands, and now even millions, of galaxies and used them to plot the three-dimensional distribution of galaxies in the Universe. The conclusion is that if we smooth the distribution on the largest scales (larger than several hundred Mpc scales), it approaches a homogeneous distribution, in accord with the assumptions that underlie the FRW model. However, as we go to smaller scales, there are density fluctuations—overdense regions like clusters and underdense regions we call voids—which increase in amplitude as we go to smaller smoothing scales. Roughly speaking, the rms density-fluctuation amplitude is over order unity if we average on ~ 10 Mpc scales.

The point of our study of large-scale structure will be to (a) determine how to describe quantitatively the distribution of mass in the Universe, (b) understand how that distribution is determined from observations, and (c) understand the origin of this structure; i.e., to develop theoretical models against which we can compare our quantitative descriptions of the observed Universe. The rules of the game is that our models for structure formation must be based on the known laws of physics, primarily gravity, and that the underlying cosmological model is an FRW Universe with cosmological parameters consistent with observations.

Fifteen years ago, this part of the class would have been a mess; there were tons of scenarios for structure formation, many of them quite nasty and/or contrived, all of which were consistent with the scant observations at the time. Now, things are different. We now have a "standard model" for the growth of structure in the Universe, a well-motivated, relatively simple and elegant theory that accounts for a huge variety of diverse observations and measurements, some of which are now quite precise. The name of the model is still up for grabs, but it usually goes by "cold-dark-matter" (CDM), ΛCDM, or something else. I prefer to call it the inflationary model for structure formation. This model assumes (all statements to be described more precisely below) (1) a nearly scale-invariant spectrum of primordial "adiabatic" density perturbations, where the term "adiabatic" means that the total density of the Universe may vary form one point to another, but that the ratio of various types of stuff (e.g., the photon-to-baryon or dark-matter–to-baryon ratios) are the same everywhere. Such primordial perturbations can be produced by inflation, a period of accelerated expansion right after the big bang. (2) Gravity is the dominant force that moves matter on the largest scales. (3) The dark matter, which constitutes $\sim 5/6$ of the nonrelativistic matter in the Universe, is composed of "cold dark matter", pressureless matter that interacts with everything else only gravitationally. The best bet is that this stuff is a WIMP, or perhaps another type of yet-undiscovered elementary particle like an axion, but it could be something else. We will see that in the CDM model, the interactions of dark matter are assumed to be purely gravitational, and this simple assumption goes a pretty long way to explaining a lot of observations. (4) The distribution of galaxies, which are regions of considerable overdensity that contain a lot of gas that can interact non-gravitationally, is related to the distribution of mass possibly through some non-gravitational physics.

2 The spherical top hat

The simplest type of perturbation is the spherical top-hat. Consider an Einstein-de-Sitter (i.e., critical-density) Universe that is perfectly homogeneous, except for one spherical region of radius r that has a slightly larger density than the mean density. Birkhoff's theorem (the relativistic generalization of Newton's theorem which states that a spherical shell exerts no gravitational force on the matter inside) says that we can treat the dynamics of this spherically overdense region by ignoring everything outside; this is exactly the same argument we used to derive the Friedmann equations. Then, the acceleration of a mass element at the surface of the sphere relative to the center of the sphere is

$$\ddot{r} = -\frac{GM}{r^2}, \tag{1}$$

where the dot denotes derivative with respect to time. This equation has a parametric solution, $r = A(1-\cos\theta)$ and $t = B(\theta - \sin\theta)$ with $A^3 = GMB^2$, which you can check by differentiating and plugging back into the equation. (The solution should look familiar, because it is the cycloid that describes the evolution of the scale factor in a closed Universe.) At early times, $\theta \to 0$, and we can expand, $\sin\theta = \theta - \theta^3/6 + \theta^5/120 - \cdots$, and $\cos\theta = 1 - \theta^2/2 + \theta^4/24 - \theta^6/720 + \cdots$. To quintic order in θ, $r = (A\theta^2/2)(1-\theta^2/12)$, and $t = (B\theta^3/6)(1-\theta^2/20)$, so $(6t/B)^{1/3} = \theta(1-\theta^2/60)$ and $\theta^2 = (6t/B)^{2/3}(1+\theta^2/30)$. Therefore,

$$r = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left(1+\frac{\theta^2}{30}\right)\left(1-\frac{\theta^2}{12}\right) = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]. \tag{2}$$

The lowest-order behavior as $t \to 0$ is $r \propto t^{2/3}$, which is just the growth of the radius of the unperturbed sphere in the Einstein-de Sitter Universe. The first correction to the unperturbed behavior results in a fractional change

$$\frac{\delta r}{r} = -\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}, \tag{3}$$

of the radius of the sphere which implies that the fractional density perturbation is

$$\delta \equiv \frac{\delta\rho}{\rho} = -3\frac{\delta r}{r} = \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}, \tag{4}$$

an approximation that is valid as long as $\theta \ll 1$, or as long as $\delta \ll 1$.

Let us now consider the further evolution of the spherical overdensity. The exact solution indicates that the sphere reaches a point of maximum expansion, known as "turnaround," when $\theta = \pi$ at radius $r = 2A$ at time $t = \pi B$. At this time, the radius the sphere would have if it had the mean (i.e., critical density) is $r = (A/2)(6t/B)^{2/3}$, so the density contrast at turnaround is

$$\frac{\rho}{\bar{\rho}} = \left[\frac{(2A)^2}{\left[\frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\right]^3}\right]^{-1} = \frac{9\pi^2}{16} \simeq 5.55. \qquad (5)$$

This occurs when the linear theory (which is no longer valid in this regime) predicts $\delta_{\rm lin} = (3/20)(6\pi)^{2/3} \simeq 1.06$. In other words, turnaround occurs when the linear-theory perturbation becomes of order unity.

After turnaround, the sphere begins to undergo gravitational collapse (just like a closed Universe in its collapsing phase), and the sphere collapses to a point at $\theta = 2\pi$ and $t = 2\pi B$, when the (invalid) linear-theory prediction is $\delta_{\rm lin} = (3/20)(12\pi)^{2/3} \simeq 1.69$. What happens next falls outside the framework of the model, but it is supposed that in a real spherical (or nearly spherical) overdensity, the gas in the spherical overdensity will shock and thus be heated, and the dark matter will undergo a poorly defined and poorly understood process known as "violent relaxation" in which the dark-matter particles get redistributed in phase space. The end result is then assumed to be a gravitationally-bound dark-matter halo with a distribution of gravitational binding energy U and kinetic energy K given by the virial theorem, $U = -2K$, and a total energy equal to the binding energy, $U = -GM/r_{\rm turnaround}$, at turnaround (when the kinetic energy is zero). If the final configuration is homogeneous, then the virialization radius $r_{\rm vir} = r_{\rm turnaround} = A$, and the radius the critical-density sphere would have at virialization $\theta = 2\pi$ is $(A/2)(12\pi)^{2/3}$. Thus, at virialization, the density contrast between the virialized sphere and the universal density is $\delta_{\rm vir} \equiv (\rho_{\rm vir}/\bar{\rho}) - 1 = (6\pi)^2/2 - 1 \simeq 177$. This analysis has assumed throughout that the outside Universe has critical density. If $\Omega_m < 1$, either because the Universe is open or because it is critical density with a cosmological constant, then the curvature or cosmological-constant terms in the Friedmann equation increase the expansion rate of the ambient Universe and so the ambient density is thus smaller at virialization. A detailed calculation is straightforward by complicated. The end result is that the virialization density can be approximated $\delta_{\rm vir} \simeq 177\,\Omega_m^{-0.7}$.

3 Linear perturbations

The spherical overdensity is instructive since it can be solved exactly, but it is quite an oversimplification. More realistically, inflation predicts that the primordial density contrast is $\delta(\vec{x}, t_i)$ with $\delta \ll 1$ at some sufficiently early initial time t_i, but is otherwise an arbitrary function of position $\vec{x}$. We will now see what happens to the density perturbation with time. To do so, we write equations of motion for the cosmological fluid. We will consider volumes of the Universe that are small compared with the Hubble distance, $|\vec{x}| \ll H^{-1}$, so that the Hubble velocities are small, $v \ll c$. In that case, we can use the usual nonrelativistic Newtonian fluid-flow equations. The first is the Euler equation,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} - \nabla \Phi, \qquad (6)$$

where $\vec{v}(\vec{x},t)$ is the velocity at position $\vec{x}$ at time t, $p(\vec{x},t)$ is the pressure, $\rho(\vec{x},t)$ is the mass density, $\Phi(\vec{x},t)$ is the gravitational potential, and $\vec{\nabla} = \partial/\partial\vec{x}$. This equation simply says that the acceleration of a fluid element is determined by the pressure gradient and the gravitational acceleration. The left-hand side involves the convective derivative $D/Dt = \partial/\partial t + \vec{v}\cdot\vec{\nabla}$. It says that the rate of change of a quantity at position $\vec{x}$ is given by the explicit change, as well as the change that occurs by the movement of fluid at that position. The gravitational potential is determined through the Poisson equation,

$$\nabla^2\Phi = 4\pi G\rho, \tag{7}$$

and there is also the continuity equation,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0. \tag{8}$$

We take as the zero-th order solution to this equation the Hubble flow, $\rho(\vec{x},t) = \rho_0(t)$ and $\vec{v}$=$H\vec{x}$. The zero-th order solution for the potential is $\Phi_0 = (2\pi G/3)\rho_0|\vec{x}|^2$. Note that here the subscripts '0' denote the zero-th order (i.e., homogeneous) solution, *not* the necessarily values today. You can check that this ρ_0, Φ_0, and $\vec{v}_0$ satisfy the Euler, continuity, and Poisson equations.

Now we consider small perturbations by writing $\vec{v} = \vec{v}_0 + \delta\vec{v}$, $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$, and $\Phi = \Phi_0 + \delta\Phi$, where the perturbations are assumed to be small compared with the zero-th order quantities. Collecting terms in the Euler equation that are first order in the perturbed quantities,

$$\frac{d\delta\vec{v}}{dt} + \frac{\dot{a}}{a}\delta\vec{v} = -\frac{\vec{\nabla}\delta p}{\rho_0} - \vec{\nabla}\delta\Phi, \tag{9}$$

where we have used $(\delta\vec{v}\cdot\vec{\nabla})\vec{v}_0 = H\delta\vec{v}$, and defined $d/dt \equiv \partial/\partial t + \vec{v}_0\cdot\vec{\nabla}$, the total time derivative for an observer comoving with the unperturbed expansion. The first-order continuity equation is

$$\frac{d\delta\rho}{dt} + \rho_0\vec{\nabla}\cdot\delta\vec{v} + 3H\delta\rho = 0, \tag{10}$$

and the Poisson equation is simply $\nabla^2\delta\Phi = 4\pi G\delta\rho$. We then define the fractional density perturbation $\delta(\vec{x},t) \equiv \delta\rho(\vec{x},t)/\rho_0(t)$, which changes the continuity equation to $\dot{\delta} = -\vec{\nabla}\cdot\delta\vec{v}$, where the dot denotes d/dt, and the Poisson equation to $\nabla^2\delta\phi = 4\pi G\rho_0\delta$. We then note further that $\delta p = c_s^2\delta\rho$, where c_s^2 is the sound speed. We then take the gradient of the (perturbed) Euler equation and replace $\vec{\nabla}\cdot\delta\vec{v}$ by $-\dot{\delta}$ and arrive at a single equation for δ,

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \left(4\pi G\rho_0 + c_s^2\nabla^2\right)\delta. \tag{11}$$

Note that the ∇^2 here is a Laplacian with respect to *physical* (not comoving) coordinate. There is a slight subtlety in arriving at this equation that arises because $\vec{x} = a(t)\vec{r}$, where $\vec{r}$ is a comoving coordinate. Because of this, $(d/dt)(\vec{\nabla}\cdot\delta\vec{v}) = (d/dt)(a^{-1}\vec{\nabla}_r\cdot\delta\vec{v}) = (\dot{a}/a)(\vec{\nabla}_r\cdot\delta\vec{v}) + a^{-1}\vec{\nabla}_r\cdot(d\delta\vec{v}/dt)$. Thus, $\vec{\nabla}\cdot(d\delta\vec{v}/dt) = \ddot{\delta} - (\dot{a}/a)\dot{\delta}$.

Now consider a single Fourier mode of the density field $\delta(\vec{x},t) = \delta_{\vec{k}}(t)\sin(\vec{k}\cdot\vec{r})$, where $\vec{k}$ is the *comoving* wavenumber of the perturbation, and $\lambda = 2\pi/k$ is the comoving wavelength of the perturbation. Then

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \left(4\pi G\rho_0 - \frac{c_s^2k^2}{a^2}\right)\delta. \tag{12}$$

In the absence of expansion (i.e., $\dot{a} = 0$), the solution to the equation is $\delta_{\vec{k}}(t) = e^{\pm t/\tau}$, where $\tau = (4\pi G\rho_0 - c_s^2 k_{\rm phys}^2)^{-1/2}$ is the growth time (and $k_{\rm phys}$ is the physical wavenumber). For perturbations with $\lambda > \lambda_J$, the Jeans length $\lambda_J = 2\pi/k_J = c_s\sqrt{\pi/G\rho_0}$, gravitational instability overcomes the pressure resistance, and the perturbations grow. If $\lambda < \lambda_J$, the perturbations are stabilized by pressure gradients, the mode amplitude oscillates with time; i.e., perturbations in the gas propagate as sound waves, just as they do when we speak in a room. In an expanding Universe, there are two differences: First, the Jeans wavelength changes with time, and secondly, the growth of the instability becomes a power law, rather than exponential, as we will now see.

Consider now the case where the pressure is negligible. We then find that linear perturbations satisfy

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\delta\rho_0 = 0. \tag{13}$$

The solution to this differential equation governs the time dependence $\delta(t)$ of the growth of linear perturbations. As an example, consider an Einstein-de Sitter (i.e., critical-density) Universe. Then, $\dot{a}/a = 2/3t$, and $4\pi G\rho_0 = (3/2)(\dot{a}/a)^2 = 2/3t^2$. The equation can be solved by plugging in the *ansatz* $\delta = At^n$, and the resulting solutions are a growing mode $\delta(t) \propto t^{2/3} \propto a(t)$ and a decaying mode $\delta(t) \propto t^{-1}$. For the Einstein-de Sitter universe (only), the potential perturbation $\delta\Phi$ remains constant in time (since $\nabla^2 \propto a^{-2}$ and $\rho_0 \propto a^{-3}$).

Now consider the growth of nonrelativistic-matter perturbations if the expansion of the Universe is driven by a combination of radiation and matter (e.g., just before decoupling). Then $(\dot{a}/a)^2 = 8\pi G(\rho_m + \rho_r)/3$, and if we let $y \equiv \rho_m/rho_r = a/a_{\rm eq}$ serve as our time variable, then we arrive at an equation,

$$\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0, \tag{14}$$

where the prime denotes derivative with respect to y. The solution to this equation is $\delta \propto y + 2/3$ which at late times is $\delta \propto a$, the matter-dominated result, but which is constant at early times, during radiation domination. The other solution to the equation is $\delta \propto \ln y$ during radiation domination. We conclude that *perturbations in the nonrelativistic-matter density grow only logarithmically until the Universe becomes matter dominated.*

It is important to realize, however, that this analysis is not, strictly speaking, relevant for CDM models. In this analysis, we have considered nonrelativistic-matter perturbations in an otherwise smooth radiation background. In CDM models, however, primordial perturbations are adiabatic, meaning that the fractional energy-density perturbation in the radiation is the same as that in the matter. The proper treatment of this situation (i.e., the consideration of relativistic perturbations) is beyond the scope of this class. The basic conclusion that perturbations do not grow during radiation domination is, however, true for CDM models, although for a different reason. As you will show in a homework problem this week, sub-horizon perturbations are stabilized during radiation domination by pressure. More specifically, you will show that the existence of pressure in the baryon-photon fluid prior to recombination leads to a Jeans length that is comparable to the horizon. Since the baryon-photon fluid dominates the energy density during radiation domination, density perturbations do not grow during this time.

Now suppose we have a cosmological constant and/or curvature with $\Omega_m \ll 1$. In this case, $4\pi G\rho_0 = (3/2)\Omega_m H^2$, and the last term is negligible resulting in an equation, $\ddot{\delta} + 2(\dot{a}/a)\delta = 0$.

This has solutions $\delta \propto$ constant and $\delta \propto a^{-1}$. Therefore, *the growth of density perturbations is slowed once the Universe becomes curvature and/or cosmological-constant dominated.*

For a model with arbitrary Ω_m and Ω_Λ, the growing-mode solution is

$$\delta_{\rm grow} \propto \frac{\dot{a}}{a} \int_0^a \frac{da}{\dot{a}^3}, \tag{15}$$

which can be verified by plugging into the differential equation. This integral can be done numerically, and can be well-approximated by the following semi-analytic fit:

$$\frac{\delta(z=0,\Omega_m,\Omega_\Lambda)}{\delta(z=0,\Omega_m=1,\Omega_\Lambda=0)} = \frac{5}{2}\left[\Omega_m^{4/7} - \Omega_\Lambda + \left(1+\frac{1}{2}\Omega_m\right)\left(1+\frac{1}{70}\Omega_\Lambda\right)\right]^{-1}. \tag{16}$$

For $\Omega_m + \Omega_\Lambda = 1$, a good approximation is

$$\frac{\delta(z=0,\Omega_m)}{\delta(z=0,\Omega_m=1)} \simeq \Omega_m^{0.23}. \tag{17}$$

4 Peculiar velocities

According to the continuity equation, $\dot{\delta} = -\vec{\nabla} \cdot \delta\vec{v}$. Therefore, the peculiar velocities induced by linear density perturbations are curl free. Moreover by re-writing,

$$\begin{aligned} \dot{\delta}(\vec{x},t) &= \delta(\vec{x},t)\frac{\dot{\delta}}{\delta} = \delta(\vec{x},t)\frac{d\delta/da}{\delta}\frac{da}{dt} && (18) \\ &= \frac{1}{a}\left(\frac{a}{\delta}\right)\left(\frac{d\delta}{da}\right) aH\delta(\vec{x},t) = f(\Omega)H\delta(\vec{x},t), && (19) \end{aligned}$$

where

$$f(\Omega) \equiv \frac{a}{\delta}\left(\frac{d\delta}{da}\right) \simeq \Omega_m^{0.6}. \tag{20}$$

Therefore,

$$\vec{\nabla} \cdot \delta\vec{v}(\vec{x},t) = aHf(\Omega_m)\delta(\vec{x},t). \tag{21}$$

Therefore, by measuring the peculiar-velocity field and the density field as a function of position $\vec{x}$, one can measure $\Omega_m^{0.6}$. This was the basis of the POTENT program which in the 1990s was a popular technique for determining Ω_m. Strictly speaking, however, the galaxy distribution that was used measures $\delta_g(\vec{x},t) = b\delta(\vec{x},t)$, where b is the "bias." In other words, the galaxy distribution is not necessarily the same as the matter distribution, and the relation between them is quantified by this fudge factor we call the bias (more below about this). Therefore, what peculiar-velocity surveys really measured was a quantity $\beta \equiv \Omega_m^{0.6}/b$, which turned out to be $\beta \simeq 0.6$. Note finally that from the continuity equation, the density perturbation is determined by the gradient of the peculiar velocity. Thus, peculiar velocities probe in some sense larger scales than do density perturbations.

5 Cosmological density fields

Inflation does not predict the primordial density field $\delta(\vec{x},t)$; it makes predictions about the statistical properties of the density field. We now discuss these statistics. The primordial density field is said to be a realization of a random field with a *correlation function*, $\xi(\vec{r}) = \langle\delta(\vec{x}+\vec{r})\delta(\vec{x})\rangle$, where the brackets denote an average over all realizations, or by the ergodic theorem, over the entire Universe. For a number of reasons which will become apparent later, it is often easier to deal with the Fourier transform of the density field,

$$\tilde{\delta}(\vec{k}) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\,\delta(\vec{x}). \tag{22}$$

Note that since $\delta(\vec{x})$ is real, $\tilde{\delta}^*(\vec{k}) = \delta(-\vec{k})$. With a bit of algebraic manipulation, it then follows that

$$\left\langle \tilde{\delta}(\vec{k})\tilde{(}\vec{k}')\right\rangle = (2\pi)^3\delta_D(\vec{k}+\vec{k}')P(k), \tag{23}$$

where $\delta_D()$ is a Dirac delta function, and the *power spectrum* $P(k)$ is a sort of Fourier transform of the correlation function,

$$P(k) = \int d^3x\,\xi(\vec{x})e^{-i\vec{k}\cdot\vec{x}} = 4\pi\int r^2\,dr\,\xi(r)\frac{\sin kr}{kr}. \tag{24}$$

The inverse transform is

$$\xi(r) = \int \frac{d^3k}{(2\pi)^3}\,P(k)e^{i\vec{k}\cdot\vec{x}} = \frac{1}{2\pi^2}\int k^2\,dk\,P(k)\frac{\sin kr}{kr}. \tag{25}$$

Determination of the matter power spectrum $P(k)$ is a very big deal in physical cosmology today. Just so you know, some authors prefer an alternative definition,

$$\Delta^2(k) \equiv \frac{1}{2\pi^2}k^3\,P(k), \tag{26}$$

which is equivalent to $d\left\langle\delta^2\right\rangle/d\ln k$, where $\left\langle\delta^2\right\rangle = \xi(0)$ is the *zero-lag* correlation function, or the variance of the mass distribution.

If the power spectrum is a power law, $P(k) \propto k^n$, then the correlation function is a power law $\xi(r) \propto r^{-\gamma}$ with $\gamma = n+3$. For galaxies $\xi_{gg}(r) \simeq (r/5\,h^{-1}\,\mathrm{Mpc})^{-1.8}$ for $1 \lesssim \xi \lesssim 10^4$, or $r \lesssim 5\,h^{-1}$ Mpc. The spectral index $n > -3$ so that $\xi \to 0$ as $r \to \infty$, and it can be shown that $n < 4$ is required if matter is composed of discrete particles. The spectral index $n = 0$ is a white-noise (i.e., no correlations) spectrum, and the $n = 1$ spectrum is a Peebles-Harrison-Zeldovich or "flat" scale-invariant spectrum. Such a spectrum is called "flat" because the power spectrum $\Delta^2_\Phi(k)$ for the potential is k-independent: i.e., $k^2\Phi_{\vec{k}} \propto \delta_{\vec{k}}$.

Strictly speaking, the power spectrum (or correlation function) can be approximated by a power law over a small range of k (or r), but it cannot be a power law for all k (or all r). To see this, note that the power spectrum $P(K) \propto \left\langle|\delta(\vec{k})|^2\right\rangle$; i.e., it is the variance of the Fourier amplitude for modes with wavenumber k. It is therefore a positive-definite quantity. Moreover, we must have $P(k) \to 0$ as $k \to 0$ if homogeneity is to be recovered on large scales. We thus find from the expression for $\xi(r)$ in terms of $P(k)$ that the correlation function satisfies the integral rule, $\int_0^\infty r^2\,dr\,\xi(r) = 0$. Therefore, if the mass is correlated ($\xi > 0$) on small scales, there must be some anticorrelation ($\xi < 0$) on large scales to compensate. The correlation function therefore cannot be a pure power law, and neither can the power spectrum.

6 The CDM power spectrum

That means that the power spectrum must be something else. We now have a huge amount of data on the power spectrum, and it agrees quite well with the so-called "CDM" power spectrum, or more precisely, that predicted by inflation. We will deal with inflation later.

To begin, we note that the comoving scale that entered the horizon at matter-radiation equality is $l_{\rm eq} = 16(\Omega_m h^2)^{-1}$ Mpc. Fourier modes of the density field with smaller wavelengths therefore entered the horizon at earlier times, during matter domination, and those with longer wavelengths entered the horizon at later times, during matter domination. Since we know that the growth of density perturbations is different during radiation domination than it is during matter domination, it stands to reason that this distance scale will play an important role in determining the power spectrum. What we will now see is that the *primordial* power spectrum $P_{\rm prim}(k)$, which inflation predicts is a nearly Peebles-Harrison-Zeldovich *primordial* power spectrum, $P_{\rm prim}(k) \propto k^n$ with $n \simeq 1$, gets "processed" by the growth of density perturbations once they enter the horizon. The end result is that the CDM power spectrum *today* is the "processed" power spectrum $P(k) = P_{\rm primordial}(k)[T(k)]^2$, where $T(k)$ is a "transfer function" that takes into account the effects of gravitational amplification of a density-perturbation mode of wavelength k.

The precise calculation of $T(k)$ is exceedingly complicated and is accomplished with numerical codes. The codes calculates for each Fourier mode $\vec{k}$ the time evolution of the density-perturbation amplitudes for baryons, photons, dark matter, and neutrinos, taking into account interactions between the photons and the baryons and also taking into account the effects of free streaming of neutrinos and also free streaming of photons near and after recombination. The growth of the gravitational potential, as well as more general metric-perturbation variables that arise in a full relativistic treatment, are also taken into account. There are now publicly available codes, such as CMBFAST and CAMB, that one can download to calculate the power spectrum for a variety of cosmological parameters, as well as for different assumptions about the primordial power spectrum, the nature of primordial perturbation (adiabatic or otherwise), and the masses of neutrinos.

The basic qualitative features of the power spectrum are, however, easy to understand. Inflation predicts that the primordial perturbations are nearly scale invariant, which means that the amplitude of each Fourier mode $\Phi_{\vec{k}}$ of the gravitational potential enters the horizon with roughly the same amplitude. We then need to recall two other facts: (1) During matter domination, $\Phi_{\vec{k}}$ (which is $\propto k^{-2}\delta_{\vec{k}}$) remains constant in time; and (2) during radiation domination, $\delta_{\vec{k}}$ remains constant with time (because of the stabilizing effects of radiation pressure, as discussed above). Since the density-perturbation amplitude (rather than the gravitational potential) remains fixed during radiation domination, modes that enter the horizon have their growth "stunted" relative to what they would be if they entered during matter domination. Since $\delta_{\vec{k}} \propto k^2\Phi_{\vec{k}}$, the small-wavelength modes that enter the horizon during radiation domination have their growth suppressed by $T(k) \simeq (k/k_{\rm eq})^{-2}$. The transfer function should thus be approximately,

$$T(k) \simeq \begin{cases} 1, & k \lesssim k_{\rm eq}, \\ (k/k_{\rm eq})^{-2}, & k \gtrsim k_{\rm eq}, \end{cases} \tag{27}$$

where $k_{\rm eq} = 2\pi/l_{\rm eq}$. Thus, if $P_{\rm primordial}(k) \propto k^n$, the processed power spectrum is $P(k) \propto k^n$ for $k \ll k_{\rm eq}$ and $P(k) \propto k^{n-4}$ for $k \gg k_{\rm eq}$.

The numerical calculations can be fit reasonably well by the following analytic form,

$$T(k) = \frac{\ln(1+2.34\ q)}{2.34\,q}\left[1+3.89\,q+(16.1\,q)^2+(5.46\,q)^3+(6.71\,q)^4\right]^{-1/4}, \tag{28}$$

where

$$q \equiv \frac{k}{\Omega_m h^2\,\mathrm{Mpc}^1}. \tag{29}$$

This functional form has the asymptotic behavior derived above, up to a logarithmic correction which is related loosely to the logarithmic growth of density perturbations during radiation domination that we discussed above. When plotted as a function of k, the CDM power spectrum $P(k)$ rises as k^n (with $n \simeq 1$) at small k, it then peaks near $k_{\rm eq}$, and then falls as $k^{n-4} \sim k^{-3}$ at large k. The value of k at which the power spectrum peaks is proportional to $\Omega_m h^2$. Usually, however, the power spectrum is plotted not as a function of k, but as a function of the observable kh (since distances scale with the Hubble parameter h). Thus, when plotted as a function of kh, the value of kh at which the power spectrum peaks scales with the parameter $\Gamma \equiv \Omega_m h$.

Neutrino damping. There is one other important thing to note. During the times just before and then after matter-radiation equality, when density perturbations begin to grow, neutrinos are free streaming; i.e., they move with the velocity of light. They can therefore transport energy/momentum from one point in the Universe to another and therefore act to smooth the density field. If, however, the neutrinos constitute a negligible fraction of the energy density of the Universe (which is true if they are massless), then the effect of free streaming on the power spectrum is negligible. If, however, the neutrino has a small mass (e.g., a few eV), then it might contribute to the energy density in a non-negligible way. If so, then free streaming of neutrinos will serve to damp the perturbations on scales smaller than their "free-streaming length", which is defined to be the distance they travel before they become nonrelativistic; i.e., the horizon distance when the temperature becomes comparable to the neutrino mass $T_\nu \simeq m_\nu$. You will calculate this distance more precisely in your homework. Roughly speaking, though, the neutrino damping length is on order 10 Mpc for $m_\nu \sim$eV; i.e., scales probed by galaxy surveys like SDSS and 2dF. Measurements of the power spectrum by these surveys can thus limit the neutrino mass to less than a few eV, a more stringent bound than the Cowsik-McLelland bound of ~ 10 eV.

7 Smoothed density fields

In practice, it is impossible to measure the density perturbation $\delta(\vec{x})$ at a particular point. Instead, it is only possible to measure the density perturbation smoothed over some volume. We thus replace the density field $\delta(\vec{x},t)$ by a smoothed field,

$$\delta_R(\vec{x},t) = \int d^3r\, W(\vec{r})\delta(\vec{x}+\vec{r}), \tag{30}$$

where $W(\vec{r})$ is a *window function*. The most common window function is the spherical top hat of radius R: $W_R(r) = 3/(4\pi R^3)$ for $r < R$ and $W(r) = 0$ for $r > R$. The Fourier transform of this particular window function is $W(k) = (3/y^3)(\sin y - y\cos y) = 3j_1(y)/y$, where $y = kR$ and $j_1(y)$ is a spherical Bessel function. The power spectrum for the smoothed density field is $P(k)|W(k)|^2$, and the variance of the mass distribution smoothed with a sphere of radius R is thus,

$$\sigma^2(R) = \frac{1}{2\pi^2} \int P(k)\,|W(k)|^2\,k^2\,dk. \tag{31}$$

This leads us to the famous σ_8 which is the variance of the mass distribution smoothed on $R = 8\,h^{-1}$ Mpc scales: i.e.,

$$\sigma_8^2 = \frac{1}{2\pi^2} \int P(k) \left[\frac{3j_1(kR)}{kR}\right]^2 k^2\,dk. \tag{32}$$

This quantity is often used to normalize the power spectrum. It turns out to be $\sigma_8 \simeq 1$ for galaxies.