Ay121

Fall 2007

RADIATIVE PROCESSES

Problem Set 5

Due in class November 8, 2007

1. Rybicki and Lightman calculate the cooling rate in a plasma due to brehmsstrahlung from electron-ion elastic scattering. Why can we neglect electron-electron scattering? Or proton-proton scattering? What about proton–alpha-particle scattering in a fully ionized plasma with 25% helium by weight? Estimate *in order of magnitude only* the cooling rates from these other processes.

2. This problem considers free-free emission from HII (ionized hydrogen) regions (the analysis can also apply to x-ray observations of galaxy clusters). Assume an HII region has a uniform electron temperature T and density n_e, which we would like to determine by observational means. The *emission measure* EM is defined to be

$$\mathrm{EM} \equiv \int n_e^2 \, ds.$$

   a. Show that the free-free optical depth in the radio regime can be written

$$\tau_\nu^{ff} = 0.018\, T^{-3/2} Z^2 \nu^{-2} \bar{g}_{ff}.$$

   b. Show that this implies that the integrated flux from the HII region is $F_\nu \propto \nu^2$ for $\nu < \nu_*$ and that F_ν is nearly flat, but slowly falling, for $\nu > \nu_*$. (You will need to use expressions from Fig. 5.2 in RL to do this latter part.)

   c. The Orion nebula has a spectrum that looks like that in part b, with $\nu_* \sim 1$ GHz and $F_{\nu_*} \sim 300$ Jy, and it subtends an half-angle of 4 arcmin at a distance of 500 pc. Estimate the temperature T and the electron density n_e. Now suppose someone tells you that the temperature is 8000 K. What is the electron density you obtain?

   d. Now suppose that the HII region is spherically symmetric and has roughly constant temperature, but is *not* uniform in density and has $n_e \propto r^{-m}$, with m some constant value. Show that the free-free emission from such a source has $F_\nu \propto \nu^\alpha$ with $\alpha = (6-4m)/(1-2m)$. What limitations on m are required for validity of this formula?

3. The Crab nebula surrounds the radio pulsar PSR 0531+21 created by the supernova of 1054 AD. The nebula has a radius of 1.8 pc (3 arcmin), and its highly polarized emission is believed by some to be due mostly to synchrotron radiation from electrons, positrons,

and magnetic field expelled by the central pulsar and entering the nebula at a termination shock of radius 0.1 pc (10 arcsec). The radio spectrum of the nebula is astonishingly uniform, $I_\nu \propto \nu^{-0.27}$ for 10^7 Hz$< 10^{13}$ Hz, with variations of spectral index of less than 0.01. The emission from $10^{12.5} - 10^{13.5}$ Hz is dominated by thermal emission from heated dust in and around the nebula. Above $10^{13.5}$ Hz, the spectrum is again dominated by synchrotron radiation, and gradually steepens from $I_\nu \propto \nu^{-0.75}$ in the optical through ν^{-1} in soft x-rays and still steeper in the hard x-ray and gamma-ray. The spectrum of the integrated light from the nebula is shown in the figure.

At frequencies above the radio, the spectral index is not constant: half of the total radio flux comes from within 100 arcsec, half of the total optical flux from within 70 arcsec, and half of the total 1 keV x-ray flux from within 40 arcsec.

a. Suppose that beginning at time $t = 0$, relativistic electrons have been injected at a constant (in time) rate $S(\gamma) \propto \gamma^{-p}$ with a power-law distribution in Lorentz factor γ; i.e., $S(\gamma)d\gamma$ is the number of electrons in the interval $\gamma \to \gamma + d\gamma$ injected per unit time. Determine the electron distribution $N(\gamma, t)$, where $N(\gamma, t)d\gamma$ is the number of electrons with Lorentz factors between γ and $\gamma + d\gamma$. Define γ_c to be the Lorentz factor for which the cooling time $t_c = \gamma/\dot{\gamma}$ equals the age T of the remnant. Show that for $\gamma < \gamma_c$ (cooling time longer than the age), $N(\gamma, T) \sim TS(\gamma)$, while for $\gamma > \gamma_c$ (cooling time shorter than the age), $N(\gamma, T) \sim t_c S(\gamma)$. Thus, show that the synchrotron spectrum $F_\nu \propto \nu^{-\alpha}$ steepens in spectral index by $\Delta\alpha = 0.5$ above $\nu_c \simeq 3Be\gamma_c^2/(4\pi m_e c)$.

b. Such a steepening ("break") occurs around 10^{13} Hz in the Crab. Use this and your result of (a) to estimate the injection-energy index p and the magnetic field B (in Gauss) in the Crab nebula.

c. Show that the total energy W_e in the electrons that produce the Crab's synchrotron radiation is

$$W_e = 3 \times 10^{48} \left(\frac{3 \times 10^{-4}\,\mathrm{G}}{B} \right)^{1.5} \mathrm{erg}.$$

Hints: Do not worry about factors of two. Show that the contributions to W_e from electrons radiating at frequency ν is $\propto \nu^{1/2} L_\nu \propto \nu^{1/2-\alpha}$, and thus from the figure is dominated by the electrons radiating in the decades around 10^{13} Hz.

d. Compare the energy W_e to the total magnetic energy $W_B = VB^2/8\pi$, where V is the volume of the Crab nebula. Find the B that minimizes $W_e + W_B$. How does this compare to the B you estimated in part (b)?

e. Using the B you estimated in part (b), estimate the energy, synchrotron lifetime, and gyroperiod of the electrons radiating at 5×10^{14} Hz (optical), 2.4×10^{17} Hz (1 keV), and 2.4×10^{21} Hz (MeV).

f. In a simple magnetically dominated magnetohydrodynamic model of the pulsar wind shock, the post-shock wind has $v \sim 0.5c(r_s/r)^2$, where $r_s = 0.1$ pc is the radius at which the pulsar wind reaches ram-pressure balance with the nebula. The post-shock

$B(r) \propto r$. Assume that $B(r)$ reaches the value you found in part (b) at 1 pc, half the nebula radius. Assume that all the electron acceleration occurs at the shock and that thereafter the electron energies are affected only by synchrotron losses. Calculate the size you expect for the x-ray emitting region, and compare it to the value quoted in the introduction. Show that the size at other frequencies should scale as $r(\nu) \propto \nu^{-1/9}$.

g. In the best current theory for acceleration at the pulsar wind shock, the electron spectrum is a relativistic Maxwellian [i.e., $S(\gamma) \propto \gamma^2 \exp(-\gamma m_e c^2/kT)$] with $kT \sim$ 500 GeV with a power-law tail $S(\gamma) \propto \gamma^{-2}$ extending to 1000 TeV. What portions of the spectrum shown in the figure can this theory explain? Be careful when thinking about the radio region of the spectrum, and be sure to consider what $N(\gamma)$ is for a Maxwellian for $\gamma m_e c^2 \ll kT$.