Steady Spherically-Symmetric (Bondi-~~Hoyle~~) accretion:

Have point mass in ~~infinite~~ homogeneous fluid of infinite extent and at rest as $r \to \infty$. Fluid accretes in spherically symmetric fashion onto star due to grav. attraction. Here we find fluid flow. The fluid satisfies the continuity (mass conservation) eqn:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{v}) = 0 \xrightarrow[\text{spherical symmetry}]{} \frac{1}{r^2}\frac{d}{dr}(r^2 \rho v) = 0$$

Mass-accretion rate is $\dot{M} = 4\pi r^2 \rho(-v)$.

Euler eqn (conservation of momentum; a fluid $F = m\vec{a}$):

$$\rho \frac{d\vec{v}}{dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}P + \vec{f},$$

is in our case, (for steady-state accretion):

$$v\frac{dv}{dr} + \frac{1}{\rho}\frac{dP}{dr} + \frac{GM}{r^2} = 0.$$

$\gamma = \frac{5}{3}$ adiabatic, $\gamma = 1$ iso-thermal

We assume the gas has EOS: $P = K\rho^\gamma$ and ideal gas law,

$$T = \mu m_H P/\rho k.$$

Some algebraic rearrangement:

$$\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = c_s^2 \frac{d\rho}{dr} \quad \text{with } c_s^2 \equiv \left(\frac{\partial P}{\partial \rho}\right)$$

$$v\frac{dv}{dr} - \frac{c_s^2}{vr^2}\frac{dv}{dr} + \frac{GM}{r^2} = 0 \quad \text{using continuity eqn}$$

$$\Longrightarrow \boxed{\frac{1}{2}\left(1 - \frac{c_s^2}{v^2}\right)\frac{d}{dr}(v^2) = -\frac{GM}{r^2}\left[1 - \left(\frac{2c_s^2 r}{GM}\right)\right]} \quad (*)$$

For $r \to \infty$, $RHS > 0$ and $\frac{d}{dr}(v^2) < 0$

$$\Longrightarrow v^2 < c_s^2 \quad \text{at} \quad r < r_s \equiv \frac{GM}{2[c_s(r_s)]^2} \quad \text{subsonic}$$

$r < \frac{GM}{2c_s^2(r_s)}$

$$r_s \cong 7.5\times10^{13}\left(\frac{T(r_s)}{10^4 K}\right)^{-1}\left(\frac{M}{M_\odot}\right) \text{cm}$$

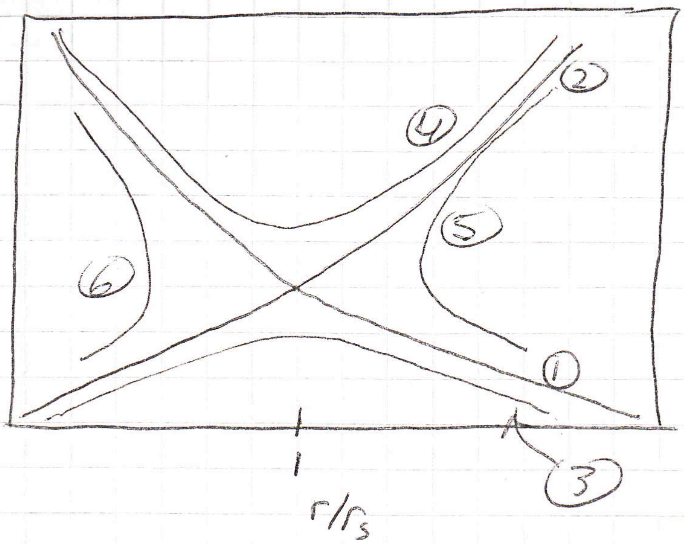
r_s is the sonic radius. At $r < r_s$ $v^2 > c_s^2$; i.e., the flow is supersonic.

Eq. (*) requires either $v^2 = c_s^2$ at $r = r_s$,

or $\frac{d}{dr}(v^2) = 0$ at $r = r_s$

Eq. (*) has solutions

$M^2 \equiv \frac{v^2}{c_s^2}$



All solns can have $v > 0$ or $v < 0$

(5) and (6) double-valued so do not provide global soln.

(2) and (4) with $v > 0$ may be stellar winds

(3) with $v > 0$ "stellar breeze"

(3) w $v < 0$ is a slowly settling atmosphere.

The Bondi soln is (1) with $v < 0$.

Note: Tiny change to fluid flow at $r \to \infty$ turns (1) into (3) or (5); thus Bondi soln is finely tuned and must be taken with grain of salt in any astrophysical application!

Integrate the Euler eqn:

$$\frac{v^2}{2} + \int \frac{dP}{\rho} - \frac{GM}{r} = \text{const}$$

and using $dP = K\gamma\rho^{\gamma-1}d\rho$ and $K\gamma\rho^{\gamma-1} = \frac{\gamma P}{\rho} = c_s^2$,

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma-1} - \frac{GM}{r} = \text{const} \quad \text{(Bernoulli integral).}$$

But $v^2 \to 0$ as $r \to \infty$ so RHS $= \frac{[c_s(\infty)]^2}{\gamma-1}$, ~~and at~~
and at r_s, $v(r_s) = c_s$, so

$$[c_s(r_s)]^2\left[\frac{1}{2} + \frac{1}{\gamma-1} - 2\right] = \frac{[c_s(\infty)]^2}{\gamma-1}$$

or $\quad c_s(r_s) = c_s(\infty)\left(\frac{2}{5-3\gamma}\right)^{1/2}$.

The mass-accretion rate is then

$$\dot{M} = 4\pi r^2\rho(-v) = 4\pi r_s^2\rho(r_s)c_s(r_s).$$

Since $c_s^2 \propto \rho^{\gamma-1}$, $\rho(r_s) = \rho(\infty)\left[\frac{c_s(r_s)}{c_s(\infty)}\right]^{\frac{2}{\gamma-1}}$

$$\Rightarrow \dot{M} = \pi G^2M^2\frac{\rho(\infty)}{c_s^3(\infty)}\underbrace{\left[\frac{2}{5-3\gamma}\right]^{(5-3\gamma)/2(\gamma-1)}}_{1 \longleftrightarrow 4.5}$$

for $\gamma = \frac{5}{3}$ $\quad\quad$ $\gamma = 1$

or $\quad \dot{M} = 1.4\times10^{11}\left(\frac{M}{M_\odot}\right)^2\left(\frac{\rho(\infty)}{10^{-24}\,g/cm^3}\right)\left(\frac{c_s(\infty)}{10^6\,cm/s}\right)^{-3}$ g/s

(or $\dot{M}c^2 = 2\times10^{31}$ erg/s - - -)

[$c_s \sim 10^6$ cm/s $= 10$ km/s for $T \sim 10^4$ K; e.g., ISM].

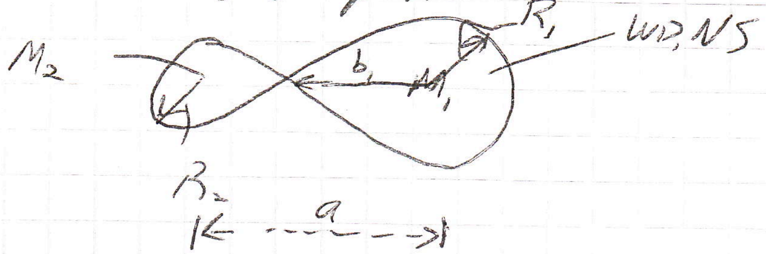
[Could have obtained this from dimensional analysis:
M's sphere of influence is $r \sim GM/c_s^2$, and
matter then flows through this surface at velocity c_s].

Outside r_s, gas is almost at rest
Inside r_s, gas approaches free-fall, $v^2 \simeq \frac{2GM}{r} = v_{ff}^2$
so
$$\rho \simeq \rho(r_s)\left(\frac{r_s}{r}\right)^{3/2} \quad \text{(from continuity).}$$
and $$T \simeq T(r_s)\left(\frac{r_s}{r}\right)^{\frac{3}{2}(\gamma-1)} \qquad r \lesssim r_s.$$

Limitations: (1) fine-tuning problem (above) $\Rightarrow$ generally expect shocks in real problems
(2) assumes no heat transport
(3) assumes no radiation, which will be important especially at small r
(4) Assumes perfect sphericity

Formation of accretion disk in compact binary:

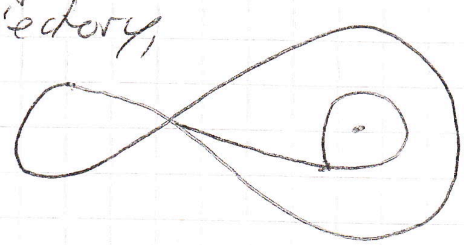
We have in mind a compact binary — e.g., low-mass X-ray binary (LMXB), a NS with MS companion ~~or~~ or cataclysmic variable (CV), a WD w. MS companion — in which compact star accretes via Roche-lobe overflow from MS companion.



In ~~an~~ non-rotating frame, $V_\perp \sim b_1\omega$ and $V_\parallel \sim c_s$, ~~i.e., esc~~ and $V_\perp >> V_\parallel$; e.g., in $\odot$, esc vel. is ~600 km/s from $\odot$ surface, and V_{orb} for Earth ~ 30 km/sec, while $V_\parallel \sim c_s$ at surface of star is $\sim\sqrt{k_B(10^4 K)/m_p}$ ~ 10 km/s. Quantitatively,

$$V_\perp \sim 100\, m_1^{1/3}(1+q)^{1/3} P_{day}^{-1/3} \text{ km/s.}$$

Thus injected matter free falls on R_1, following a trajectory,



Stream self-intersects, heats, while maintaining ang. momentum, to relax to circular orbit at circularization radius

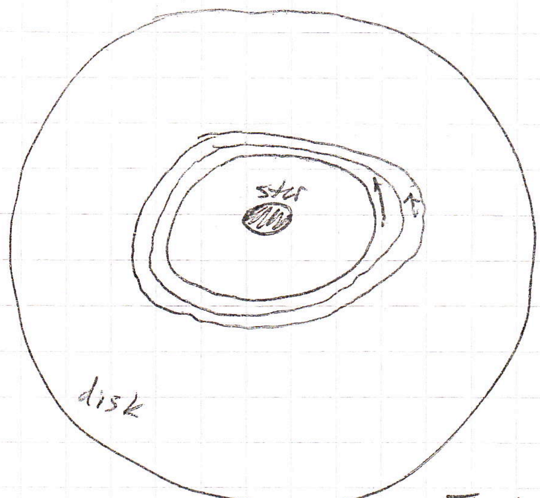
$$R_{circ} V_\phi(R_{circ}) = b_1^2\omega \quad \text{w.} \quad V_\phi(R_{circ}) = \left(\frac{GM_1}{R_{circ}}\right)^{1/2}$$

Numerically, $R_{circ} \simeq 4(1+q)^{4/3}[0.5-0.227\log q]^4 P_{day}^{2/3} R_\odot$

Typically, $R_\odot \sim R_{circ} < R_1$

Thus, R_c well inside Roche lobe of accreting star.
If accreting star is MS, then $R_{circ} \lesssim R_*$, and accreting stream just hits stellar surface (e.g., Algol).

But in WD/NS, $R_{circ} \gg R_{WD}, R_{NS}$; then pressure, viscous torques spread out circularized stream into accretion disk around star.



Consider two adjacent annuli:

Orbits of fluid elements at radius R have Keplerian Ω, i.e.

$$\Omega_K(R) = \left(\frac{GM_1}{R^3}\right)^{1/2}.$$

$\therefore$ Fluid at slightly larger radius drags fluid at slightly smaller by viscous coupling.
Fluid at smaller R then decelerated, loses ang mom $\Rightarrow$ is driven to smaller R
While Fluid at larger R accelerated, driven to higher ang mom $\Rightarrow$ larger R

$\Rightarrow$ **Viscosity transports angular momentum outward and mass inward in accretion disk!**

Viscosity heats gas $\Rightarrow$ radiation.

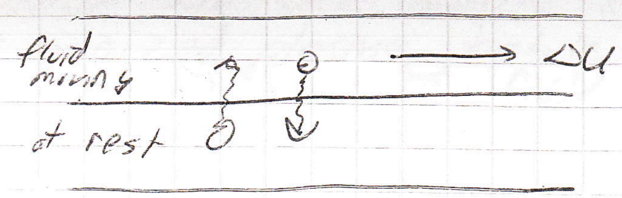
If R_{in} is ~~R at~~ inner R of accretion disk and $R_{in} \gg R_{outer}$, then grav binding energy ~~of~~ (per unit mass) of fluid element that has reached R_{in} is
$\frac{1}{2}\frac{GM_1}{R_{in}}$.

$\therefore$ If accretion rate is $\dot{M}$, disk luminosity (driven by release of grav binding energy, as fluid element moves inward is

$$\boxed{L_{disk} = \frac{GM_1\dot{M}}{2R_{in}} = \frac{1}{2}L_{acc}}$$

where $L_{acc} \equiv GM_1\dot{M}/R_{in}$

Viscosity: is therefore essential for accretion disk to work. Consider two adjacent fluid layers with velocity difference Δu:



$x \rightarrow$ Will have molecules moving up and down with velocities $V_\perp$; these transport x component of fluid momentum distance λ with velocity $V_\perp$.

Quantified by kinematic viscosity $\nu \sim \lambda v$, where λ is molecular ~~mean-free~~ mean-free path, and v is typical molecular velocity, $v \sim c_s$.

The viscous dissipation rate ($\frac{\text{energy}}{\text{area}}$ due to viscous heating) in accretion disk is ~~(dimensionally)~~

$$D(R) = \frac{9}{8}\nu \Sigma \frac{GM}{R^3},$$

where Σ is the mass per area.

We might guess that ν is ~~the~~ due to molecular interactions in the gas. ν has units of ang. momentum. We can estimate its effects therefore by comparing with orbital ang momentum via the Reynolds #:

$$Re = \frac{R V_\varphi}{\nu}.$$

If we plug in ν values for $\nu \sim \lambda v$, will have $v \sim V_\varphi$, but $\lambda \ll R \Rightarrow Re \gg 1$ (e.g., $\sim 10^{14}$).

What this implies is that molecular viscosity is <u>way</u> too small: e.g., would require $\sim 10^{14}$ orbits ($\sim 10^{7}$ yrs) for mass to spiral in. This would require disk mass

$$M \sim (\dot{M}) 10^7 \, yr \sim 10^{-8} \frac{M_\odot}{yrs} 10^7 \, yr$$

$$\sim M_\odot$$

Better understood either in hindsight, or by knowing that in laboratory setting, if $Re \gtrsim 1000$ or so, turbulence sets in. Molecular motions then replaced by random bulk fluid motions of gas so that $Re \sim 100$-1000
In ionized plasma, turbulence may be enhanced/augmented or accelerated by $\vec{B}$ fields:

$\Rightarrow$ <u>Shakura-Sunyaev α prescription</u> (1973):

We simply parameterize $\nu \equiv \alpha c_s H$, with H = height of disk.

Will see later that $\alpha \sim 10^{-3}$–1 $\gg$ molecular.

<u>The Standard Thin (Shakura-Sunyaev; Novikov-Thorne) disk:</u>

Assumes fluid confined to disk of height $H \ll R$, and no fluid flow in z direction. ~~Fluid eqns express~~ Also assumes disks are optically thick.
Fluid eqns express conservation of mass and ang. mom.

One of the results is $\nu \Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$.

Using $D(R) = \frac{9}{8} \nu \Sigma \frac{GM}{R^3}$ and noting a

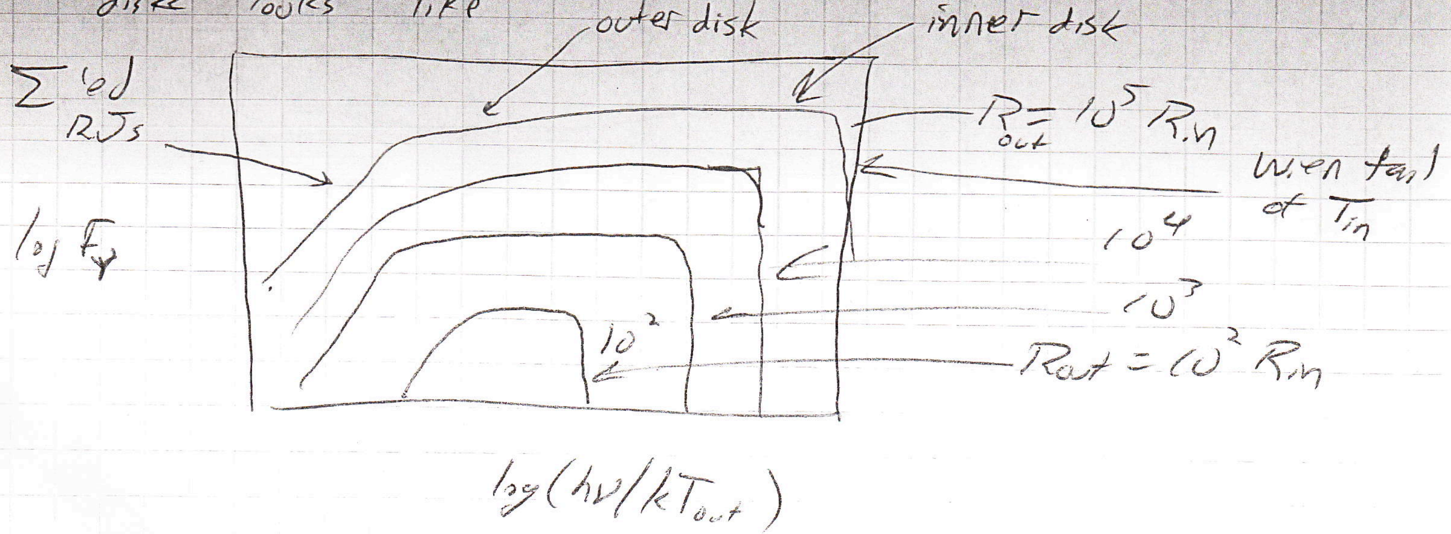
surface temperature $\sigma T^4 = D(R)$, we find that the emitted spectrum of the disk is

$$T(R) = \left\{ \frac{3GM\dot{M}}{8\pi R^3 \sigma}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]\right\}^{1/4}$$

$$\sim T_* \left(\frac{R}{R_*}\right)^{-3/4} \quad \text{for } R \gg R_*$$

and $T_* \sim \begin{cases} 4 \times 10^4 \text{ K} & \text{WD} \\ 10^7 \text{ K} & \text{NS} \end{cases}$

The emitted spectrum ~~looks like~~ integrated over the entire disk looks like



i.e., are Σ's over blackbodies

<u>The Shakura-Sunyaev solution is.</u>

(assumes Kramer's opacity, $P = \rho k T / m_p$)

~~④ $\rho = \Sigma / H$~~

$$\Sigma = 5.2\, \alpha^{-4/5}\, \dot{M}_{16}^{7/10}\, m_1^{1/4}\, R_{10}^{-3/4}\, f^{14/5}\ \mathrm{g/cm^2}$$

$$H = 1.7\times10^{8}\, \alpha^{-1/10}\, \dot{M}_{16}^{3/20}\, m_1^{-3/8}\, R_{10}^{9/8}\, f^{3/5}\ \mathrm{cm}$$

$$\rho = 3.1\times10^{-8}\, \alpha^{-7/10}\, \dot{M}_{16}^{11/20}\, m_1^{5/8}\, R_{10}^{-15/8}\, f^{11/5}\ \mathrm{g/cm^3}$$

$$T_c = 1.4\times10^{4}\, \alpha^{-1/5}\, \dot{M}_{16}^{3/10}\, m_1^{1/4}\, R_{10}^{-3/4}\, f^{6/5}\ \mathrm{K}$$

$$\tau = 190\, \alpha^{-4/5}\, \dot{M}_{16}^{1/5}\, f^{4/5}$$

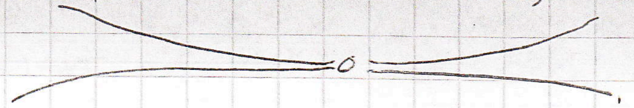
$$\nu = 1.8\times10^{14}\, \alpha^{4/5}\, \dot{M}_{16}^{3/10}\, m_1^{-1/4}\, R_{10}^{3/4}\, f^{6/5}\ \mathrm{cm^2/sec}$$

$$V_R = 2.7\times10^{4}\, \alpha^{4/5}\, \dot{M}_{16}^{3/10}\, m_1^{-1/4}\, R_{10}^{-1/4}\, f^{-14/5}\ \mathrm{cm/sec}$$

$$f = \left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]^{1/4}.$$

A few things:

① $\frac{dH}{dR} \propto R^{1/8}$ at $R > R_*$, so disks are flared,



and outer disk may be irradiated ~~at~~ by central source.

② ~~Subbs~~ Assumptions of SS soln may be inconsistent with soln; e.g., if define critical accretion rate,

$$\dot{M}_{crit} = \frac{L_{edd} R_*}{2\eta G M} = 1.5\times10^{18}\left(\frac{R_*}{3\,km}\right)\left(\frac{\eta}{0.1}\right)^{-1} g/sec$$

required for $L_{edd} = \frac{4\pi G M m_p c}{\sigma_T}$, assuming

radiation efficiency $\eta \approx 0.1$, then

$$\frac{H}{R_*} \simeq \frac{3}{4\eta}\frac{\dot{M}}{\dot{M}_{crit}}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right] \gtrsim 1$$

for $\dot{M} \to \dot{M}_{crit}$; i.e., thin-disk approximation may break down near central object.

③ Thermal stability: If $T_c \gtrsim 10^4 K$ and disk is optically thin, cool areas may cool more rapidly than can be heated leading to large T flucts (~ two-phase instability in ISM) $\Rightarrow$ SS soln not reliable

④ Viscous stability:

~~If~~ Steady disc flow stable only if $\left(\frac{\partial(\nu\Sigma)}{\partial\Sigma}\right) > 0$; otherwise, disk breaks up into rings.

~~Stability occurs if~~ Using $T \propto \Sigma^{\frac{13-2n}{4(7-2n)}}$,

instability if $\frac{7}{2} < n < \frac{13}{2}$ (when $\frac{\partial T}{\partial \Sigma} < 0$).

occurs near $T \sim 6500\,K$

Also have $\frac{\partial T}{\partial \Sigma} < 0$ (viscous instability)
if disk pressure radiation dominated and $\kappa = \kappa_{Th}$.

This may occur in inner regions of disks around centers of NS's and BHs.

<u>More general accretion flows:</u>

The standard SS (thin-~~disk~~) accretion disk assumes
① viscous energy radiated where it is produced in disk;
② gas-pressure support; and ③ purely toroidal orbits (no ~~pb~~ poloidal motions).

① If fluid can carry thermal energy inward (advect it), rather than radiate it immediately, then the heating rate per unit volume is:

$$q_{adv} = \rho V_R T \frac{ds}{dR} = \underbrace{q_+}_{\text{viscous heating rate}} - \underbrace{q_-}_{\text{radiative cooling rate}}$$

Three possibilities (limiting cases):

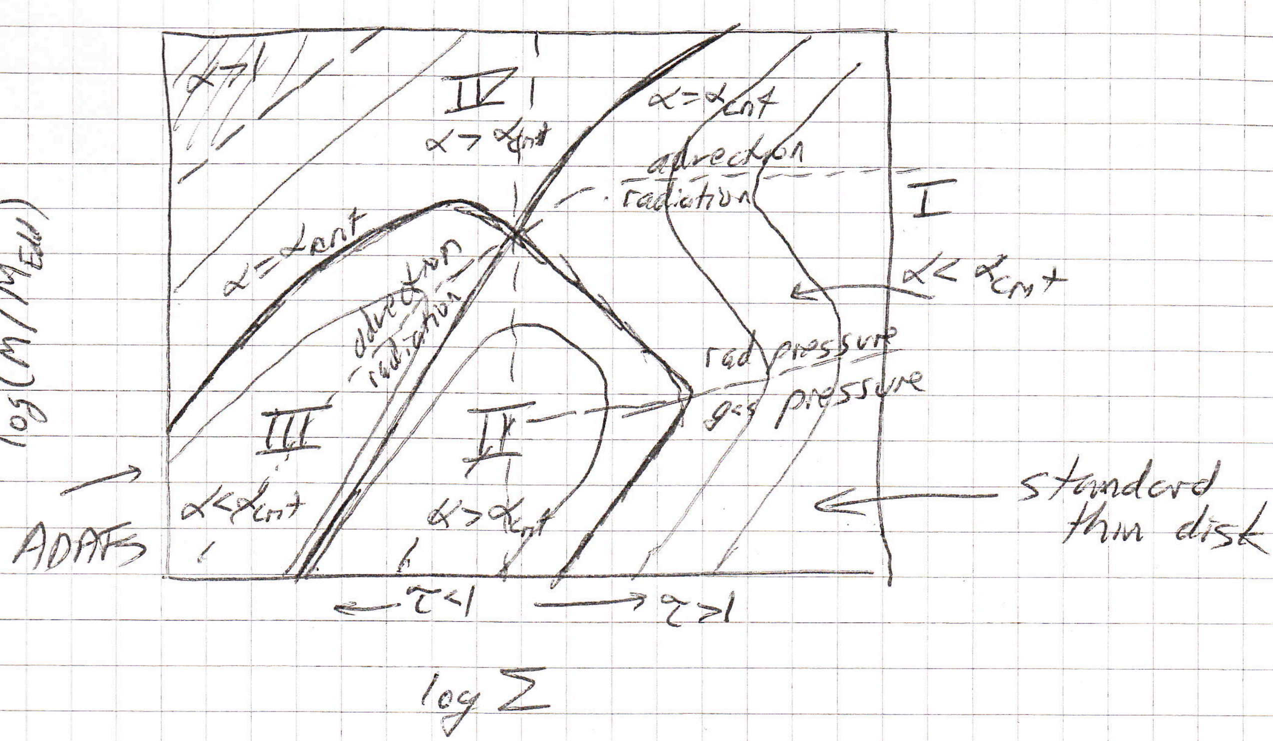
(i) $q_{adv} \ll q_+ \simeq q_-$ SS disk

(ii) $q_- \ll q_{adv} \simeq q_+$ advection-dominated accretion flows (ADAFs).

(iii) $q_+ \ll q_- \simeq -q_{adv}$ cooling flows (e.g. in galaxy clusters).

If have ADAF around BH, may simply advect viscously heated fluid into BH. $\Longrightarrow$ low radiative efficiency $\Rightarrow$ allows high $\dot{M}$ for given L.

The possible solutions can be shown on a diagram



① $\exists$ two soln. curves for each value of α

② $\exists$ multiple accretion flows for same $\dot{M}$ at different Σ

③ Similarly for same Σ

④ Standard thin disk are $\dot{M} \propto \Sigma^{5/3}$ solns at bottom right (gas-pressure dominated; radiative. and only These exist for $\alpha > \alpha_{crit}$ and $\alpha < \alpha_{crit}$; below some $\dot{M}_{max}$

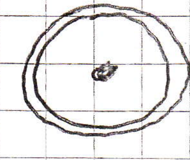
⑤ Accretion-dominated flows have $\dot{M} \propto \Sigma$;
at high $\dot{M}$, are optically thick (slim discs) $\leftarrow \forall\ \alpha$
low $\dot{M}$ are optically thin (ADAFs) $\alpha < \alpha_{crit}$ only

⑥ Optically thin, gas pressure dominated, radiatively cooled $\dot{M} \propto \Sigma^2$

⑦ Lightman-Eardley: radiation pressure, $\tau > 1$, $\dot{M} \propto \Sigma^{-1}$

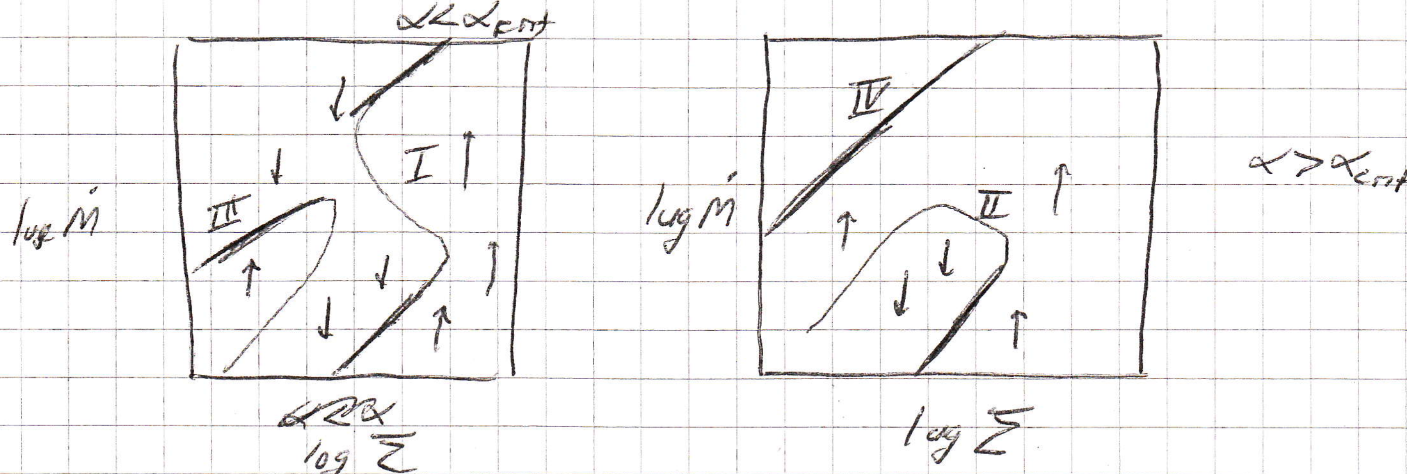
(8) Branches with $d\log \dot{M}/d\log \Sigma < 0$ undergo viscous instability to formation of rings.

Consider a thin annulus: and suppose Σ increased therein.



If $d\log \dot{M}/d\log \Sigma < 0$, then $\dot{M}$ in that annulus decreases, leading to even further increase in Σ $\Rightarrow$ runaway growth of Σ.

(9) Can also show that LE solutions have thermal instability.



arrows indicate evolution of soln in which heating not balanced exactly by (cooling + advection).

ADAFs: (or RIAFs (Radiatively-inefficient accretion flows):

Advection occurs because e^--ion coupling is weak; energy from p not transferred to e^-'s (which do bulk of radiation). I.e, radiative cooling does not occur.

M87 jets low L

also for sub-luminous AGN

Applications:

(1) Sgr A* (Galactic center)

$M_{BH} \sim 3.4 \times 10^6 M_\odot$, but $L \ll L_{Edd}$.

Could simply guess $\dot{M}$ small, but ADAF allows observed L w/ $\dot{M} \sim 10^{-5} \alpha M_\odot/yr$ and is claimed to fit spectrum better than thin disk

② Soft x-ray transients:

Binaries w. either NS or BH.
Quiescent-state ~~bet~~ spectrum better fit ?
by ADAF than thin disk.
Is argued (but controversial) that when compact * has $M \leq M^{ns}_{ch}$, see also hard x-rays from material hitting NS.

Caveats/Problems:

① Is highly unlikely that e^-'s stay cool. Collective effects in plasmas generate $\vec{B}$ fields that then very rapidly couple e^-'s + p's.

② In ADAF, particles don't radiate ⟹ are therefore only marginally (gravitationally) bound. Are therefore very easily blown off as winds

⟹ ADIOS models (adiabatic inflow-outflow solns) incorporate mass loss to winds

Every p that goes into BH liberates enough energy to blow off billions.

③ More realistically, all these models are academic; simulations show strong $\vec{B}$-field effects, turbulence, convection, winds, etc.