High-Energy Astrophysics (Ay125), Spring 2009

Problem Set 1

Due: In class, 9 April 2009

1. **White-dwarf mass-radius relation.** In this problem you will numerically integrate the equation of hydrostatic equilibrium to obtain the mass-radius relation for white dwarfs.

   (a) Begin by showing that if the pressure in the white dwarf is provided entirely by electron degeneracy pressure, then the pressure is

   $$P = \frac{8\pi m_e^4 c^5}{3h^3} \int_0^{x_F} \frac{x^4}{(1+x^2)^{1/2}} dx. \qquad (1)$$

   Here, x_F is the dimensionless Fermi momentum,

   $$x_F \equiv \frac{p_F}{m_e c} = \left[\frac{3Y_e \rho}{8\pi m_H}\right]^{1/3} \frac{h}{m_e c}, \qquad (2)$$

   where ρ is the mass density and Y_e the number of electrons per nucleon.

   (b) Show that

   $$\frac{dP}{d\rho} = Y_e \frac{m_e c^2}{3m_H} \frac{x_F^2}{(1+x_F^2)^{1/2}}. \qquad (3)$$

   (c) Now devise a numerical scheme to integrate the two coupled differential equations,

   $$\frac{d\rho}{dr} = \frac{d\rho}{dP}\frac{dP}{dr}, \qquad \text{and} \qquad \frac{dm(r)}{dr} = 4\pi r^2 \rho(r), \qquad (4)$$

   using the equation of hydrostatic equilibrium for dP/dr. You can use Mathematica, write a C or Fortran program, or any other numerical procedure you find handy.

   (d) As you change the central density, you should get a one-parameter family of stars with a sequence of masses M and radii R. You should be able to understand the qualitative behavior of your results for $R \to 0$ and $R \to \infty$.

   (e) Plot this mass-radius relation for $\mu_e = 56/26$ for an iron white dwarf and $\mu_e = 2$ for a carbon white dwarf. Plot the following three white dwarfs on your graph: (1) Sirius B, $M = 1.053\,M_\odot$, $R = 0.0074\,R_\odot$; (2) 40 Eri B, $M = 0.48\,M_\odot$, $R = 0.0124\,R_\odot$; (3) Stein 2051, $M = 0.50\,M_\odot$ or $0.72\,M_\odot$, $R = 0.0115\,R_\odot$, and try to infer their compositions.

(f) The results you have derived above should show that as $M \to 0$, $R \to \infty$. Clearly, at some point this result must break down (think about where Jupiter would fall on this plot!). This is because when the density becomes sufficiently low, Coulomb interactions between the electrons and ions (neglected above) become important in determining the equation of state. Estimate the Coulomb energy per electron. Use this result to estimate the density at which Coulomb effects become important. You can then estimate (very roughly) the maximum radius for a white dwarf and the mass at which it occurs.

2. **Diffusion of Elements.** A few white dwarfs have been found to have metals in their atmospheres. In this problem you will understand why this is strange.

   (a) What is the macroscopic force on an ion with charge Ze and mass Am_u in a star otherwise composed of pure hydrogen? (Don't forget the electric field $eE = m_p g/2$ that has to be present to keep the protons from sinking and the electrons from flying out of the star.)

   (b) Roughly how often does an ion collide with the electrons and protons in the gas? Which collisions determine the rate of downward settling? Estimate numerical values for ions in the atmosphere of an $0.6\,M_\odot$ white dwarf.

   (c) Combining (a) and (b), calculate the rate of downward drift of an ion through the white-dwarf atmosphere as a function of the density, temperature, and gravity in the gas. How does the time to drift through the atmosphere scale with the temperature of the atmosphere (scale to a fiducial value of 10^4 K)? Should we see heavy elements in the white-dwarf atmosphere? or just pure hydrogen?

3. **White-dwarf oscillations.** White dwarfs are observed to undergo oscillations with periods (very roughly) of ~ 500 sec. Using typical white-dwarf numbers (e.g., carbon interior, mass $M = 0.6\,M_\odot$ and radius $R = 0.01\,R_\odot$), to show that the the Brunt-Vaisala frequency in the outer envelope of the white dwarf is roughly consistent with the observed g-mode frequency. How does this result depend on the internal temperature?