Cosmic Rays: II. Acceleration and Propagation
Thursday, March 4, 2010

CONTENTS:

---

1. Introduction

We will now consider the acceleration and propagation of cosmic rays. These are not as well understood as the energy loss mechanisms, however they are critical portions of the subject.

References:

Our discussion of cosmic ray acceleration is an oversimplification of Krolik, *Active Galactic Nuclei*, §8.6.2.2.

Turbulent reacceleration is covered in Seo & Ptuskin 1994, ApJ 431, 705.

2. Acceleration

The acceleration of cosmic rays is believed to take place via **Fermi acceleration**, which utilizes converging velocity flows (primarily in shocks). It is also possible for cosmic rays to acquire additional energy in the ISM by interaction with Alfvén waves. We will discuss those processes briefly here.

A. SHOCK ACCELERATION

The mechanism makes critical use of the fact that the fluid flow at a shock is converging, div $\mathbf{v} < 0$, and that the fastest-moving charged particles in the thermal energy distribution suffer little collisional energy loss in the time it takes to cross the shock. (Remember the v^{-2} scaling.)

At a quantitative level, we may understand particle acceleration in a shock by tracking its position $\mathbf{x}$ and momentum $\mathbf{p}$. We will work in the shock frame so that the phase space density may be taken as time-steady; but we must occasionally make use of the fluid frame, where momenta will be primed: **p'**. For fluid velocity u (assumed in the z-direction), and assuming u is small compared with the CR velocity (usually a good assumption), these are related by:

$$\mathbf{p}'_\perp = \mathbf{p}_\perp,$$

$$p'_z = p_z - \frac{u}{c}\sqrt{p^2+m^2c^2},$$

$$p' = p - \frac{u}{c}\sqrt{p^2+m^2c^2}\cos\theta,$$

$$\cot\theta' = \cot\theta - \frac{u}{c}\frac{\sqrt{p^2+m^2c^2}}{p}\csc\theta = \cot\theta - \frac{u}{v}\csc\theta.$$

Here θ is the angle between the CR propagation direction and the z-axis. The fluid velocity is u_1 pre-shock ($z<0$) and u_2 post-shock ($z>0$).

Now we imagine an ensemble of high-energy particles that are capable of crossing the shock many times, and on either side they are capable of scattering off of inhomogeneities in the magnetic field. On either side of the shock, this scattering tends to isotropize the particles. When a particle crosses the shock, there is a discrete change in p' given by (recalling that p is conserved):

$$\Delta p' = \frac{u_1 - u_2}{c}\sqrt{p^2 + m^2c^2},$$

at normal incidence. This is the change in momentum associated with passage of a shock on the x, y, and z axes so we will divide this result by 3. Therefore we obtain a gas-frame momentum change of:

$$\Delta p' = \frac{u_1 - u_2}{3c}\sqrt{p^2 + m^2c^2} \approx \frac{u_1 - u_2}{3c}\sqrt{p'^2 + m^2} = \frac{u_1 - u_2}{3v'}p'.$$

Here v' is the velocity of the CR in the gas frame. This is doubled if we count both the upward and downward crossings.

So we observe that a CR sees an increase in its momentum as it crosses the shock, which is a good thing if we are trying to get acceleration. But it is not enough: we can get changes to p' that are only of order the shock velocity divided by the CR velocity, which is very small if the CR is supposed to be accelerated to relativistic speeds. The solution is for the CR to cross the shock many times, picking up a momentum boost at each step. Therefore we need to know how many times a CR will cross the shock. If the CR is upstream then it is clear that in the presence of scattering it will eventually be advected back into the shock, so the question is the loss rate of CRs downstream.

The loss rate is most easily calculated using phase space conservation arguments. CRs that cross the shock traveling downstream must have $\theta<\pi/2$. In the downstream frame, we see that this corresponds to $\theta'<\pi/2+u_2/v'$. The particles that travel back upstream and cross the shock have $\theta' > \pi/2 + u_2/v'$. It is thus clear that some particles are lost: if the phase space density postshock is isotropic, the ratio of upstream to downstream-propagating particles is:

$$\frac{N_{\rm up}}{N_{\rm down}} = \frac{1+\cos\theta'_{\rm max,up}}{1-\cos\theta'_{\rm min,down}} = \frac{1-u_2/v'}{1+u_2/v'} \approx 1 - 2\frac{u_2}{v'},$$

so there is a probability of $\sim 2u_2/v'$ of losing the particle downstream. This is small, which is good: a particle can cross the shock many times before it is "lost."

We can now see that in order to accelerate the particle by a fractional amount ε ($p' \rightarrow p'(1+\varepsilon)$), we incur a loss probability of

$$P_{\rm Loss} = \frac{\varepsilon}{(2/3)(u_1-u_2)/v'}\frac{2u_2}{v'} = 3\varepsilon\frac{u_2}{u_1-u_2}.$$

Defining the shock compression ratio $R = u_1/u_2$, we simplify this to:

$$P_{\rm Loss} = 3\varepsilon\frac{1}{R-1}.$$

This means that the probability of accelerating the particle to a momentum $>p$ is

$$P(>p) = \left(\frac{p}{p_0}\right)^{-3/(R-1)}.$$

Despite the crudeness of our approach, the exponent is borne out by more accurate calculations.

For a strong shock, R=4, so we expect that accelerating a particle to momentum p has a probability proportional to $1/p$. Thus we see that particles can be accelerated from low to very high energies. The nature of the "seeding" process (e.g. how a few particles from the tail of the thermal momentum distribution are injected into the above acceleration process) is not so well understood.

The above mechanism explains some features of the CR distribution:

- The predicted power law spectrum for protons is $dP/dp \sim p^{-2}$. This is shallower than observed ($\sim p^{-2.7}$) which indicates either propagation effects (e.g. cooling, leakage out of the Galaxy) or incompleteness of the acceleration model.
- The electrons cool in a time proportional to p^{-1}. This means their overall power law spectrum should be closer to $dP/dp \sim p^{-3}$. The amount of energy per logarithmic range in electron energy is constant. This is roughly consistent with observations of synchrotron radiation in our Galaxy ($\nu I_\nu \sim$ constant).

- The protons start at a higher momentum than the electrons so they are more easily accelerated.

B. TURBULENT REACCELERATION

In addition to shocks, it is possible for cosmic rays to be reaccelerated in the general ISM. The idea is that a CR travelling through a medium containing Alfvén waves can undergo changes in its energy associated with the electric fields from the Alfvén wave. Specifically,

$$\Delta p = \int \mathbf{E} \cdot \hat{\mathbf{v}} dt,$$

where $\hat{\mathbf{v}}$ is the direction of the velocity vector. The electric field is determined by the fluid motion and the magnetic field,

$$\Delta p = -\int (\mathbf{B} \times \mathbf{u}) \cdot \hat{\mathbf{v}} dt.$$

The fluid motions are critical: one cannot change the CR momentum magnitude by scattering from stationary features.

Statistically, Δp can be either positive or negative. We may however, evaluate its rms:

$$\left\langle \Delta p^2 \right\rangle = \int \int [(\mathbf{B} \times \mathbf{u}) \cdot \hat{\mathbf{v}}](t)[(\mathbf{B} \times \mathbf{u}) \cdot \hat{\mathbf{v}}](t')dtdt',$$

or we may write its **momentum diffusion coefficient**,

$$D_{pp} = \frac{\left\langle \Delta p^2 \right\rangle}{\Delta t} = \int \left\langle [(\mathbf{B} \times \mathbf{u}) \cdot \hat{\mathbf{v}}](t)[(\mathbf{B} \times \mathbf{u}) \cdot \hat{\mathbf{v}}](t+\tau) \right\rangle d\tau.$$

Thus the momentum changes by an amount $\sim(D_{pp}\Delta t)^{1/2}$ in time Δt. The diffusion coefficient is quite complicated to evaluate, seeing as it depends on the correlation function of the fluid velocity **u** at different times (or, more accurately, at different places, since the integrand is evaluated along the CR trajectory). If one works in the limit of small waves on a background field, then one may treat **B** and $\hat{\mathbf{v}}$ by their background (unperturbed) values, but the computation is still messy.

A more interesting aspect of momentum diffusion is that there is a net tendency to increase the momentum. You will derive this on the homework; the basic idea is that conservation of phase space density implies that any rearranging of particle momenta must have the net effect of taking particles confined to low-p regions of phase space (which have a phase-space volume of $4\pi p_{\max}^3 V/3$) and moving them; and such redistribution must move some of them to higher-momentum states.

3. Spatial Transport

We are finally in a position to investigate the spatial diffusion of CRs. Clearly in a uniform magnetic field, CRs would simply spiral around on helical trajectories forever. But in real life, they scatter off of features in the magnetic field and hence their spatial transport is limited, even though they do not undergo physical "collisions."

A. PITCH ANGLE SCATTERING

We consider a CR traveling in a medium with a nearly uniform magnetic field but containing Alfvén waves that are propagating "upward." (We will consider downward-propagating waves later.) We take the CR to have momentum p and **pitch angle** (direction relative to the magnetic field) ϑ. Then in a reference frame that is moving upward at the Alfvén velocity, the CR has energy:

$$E_{\uparrow} = c\sqrt{p^2 + m^2c^2} + v_{\mathrm{A}}p\cos\vartheta.$$

Since the magnetic field structure in this frame is fixed, there is no electric field in the upward-moving frame and $E_{\uparrow}$ is conserved. Therefore in scattering off an Alfvén wave, the momentum and pitch angle must change in accordance with:

$$\Delta E_{\uparrow} = \left(\frac{2cp}{\sqrt{p^2 + m^2c^2}} + v_{\mathrm{A}}\cos\vartheta\right)\Delta p - v_{\mathrm{A}}p\sin\vartheta\Delta\vartheta = 0$$

The $v_{\mathrm{A}}\cos\vartheta$ term is negligible in most cases, so we may simplify this to:

$$\Delta\vartheta = \frac{2c}{v_A\sqrt{p^2 + m^2c^2}\ \sin\vartheta}\Delta p.$$

Therefore, the changes in momentum mentioned in the section on diffusive acceleration also imply changes in the pitch angle. The diffusion coefficients are:

$$D_{\vartheta\vartheta} = \frac{\langle\Delta\vartheta^2\rangle}{\Delta t} = \left(\frac{2c}{v_A\sqrt{p^2 + m^2c^2}\ \sin\vartheta}\right)^2 D_{pp}.$$

This applies to downward-propagating Alfvén waves as well, and the diffusion coefficients still satisfy the above relation so long as the upward and downward-going waves are uncorrelated.

The above diffusion coefficient implies that ϑ changes by of order unity in a time:

$$t_{\rm scat} \sim D_{\vartheta\vartheta}^{-1} \sim \left(\frac{v_A\sqrt{p^2+m^2c^2}}{c}\right)^2 D_{pp}^{-1}.$$

This implies that in this time the direction of propagation of the CR can be completely changed, *without any actual collisions*. Thus the CR scatters off of the inhomogeneities in the magnetic field. It has a spatial diffusion coefficient (cm^2/s) given by

$$D_{xx} \sim \frac{\langle \Delta x^2\rangle}{\Delta t} \sim v^2 t_{\rm scat} \sim \left(\frac{cp}{\sqrt{p^2+m^2c^2}}\right)^2\left(\frac{v_A\sqrt{p^2+m^2c^2}}{c}\right)^2 D_{pp}^{-1} \sim v_A^2 p^2 D_{pp}^{-1}.$$

Thus, we come to the result that *the spatial and momentum diffusion coefficients are inversely related*.

The estimated diffusion coefficient based on assuming a turbulent spectrum in the ISM is[1]

$$D_{xx} = 3\times10^{28}\,\frac{v}{c}\left(\frac{R}{7\mathrm{GeV}/e}\right)^{1/3}\mathrm{cm}^2/\mathrm{s} = 0.1\frac{v}{c}\left(\frac{R}{7\mathrm{GeV}/e}\right)^{1/3}\mathrm{pc}^2/\mathrm{yr},$$

where R is the rigidity. We can thus see that for a galactic disk of thickness $\sim$ 1 kpc, the diffusion timescale should be $\sim 10^7$ yr.

B. HIGH-RIGIDITY PARTICLES

There is a limit to the energy of CRs that may be trapped in the Galaxy. This arises by considering the Larmor radius in a magnetic field of $\sim$6 μG:

$$r_{\rm L} = 0.2\frac{R}{\mathrm{PeV}/e}\,\mathrm{pc}.$$

For particles with rigidity exceeding $\sim 10^{17}$ eV/e, it follows that the Larmor radius becomes comparable to the thickness of the Galactic thin disk. For the ultra-high-energy cosmic rays, with rigidity of $\sim 10^{19}$ eV/e, the Larmor radius is several kpc. Such particles cannot be confined within our Galaxy and must arise from extragalactic sources.

---

[1] Many references; see e.g. Berezinsky 1990 (Proc. 21st Intl. Cosmic Ray Conference, 11, 115)