19 Jan 2010 **Ay 126** E.S. Phinney & W.L.W. Sargent

The Interstellar Medium

Problem Set 3

Due **in class** Thursday 28 Jan, 2010

Readings: Chapter 7 of Tielens.

Homework Problems: Note: problems 3 and 4 are not nearly as long as they look: I wrote a lot of words to lead you through the concepts, but the actual solutions are only a couple of pages.

1. **A recombination paradox** [10 points]

   A fully ionized gas which starts to radiate recombination radiation heats up. Explain this paradoxical result.

2. **Dust in HII regions** [10 points]

   Consider an HII region of density $n_H = 10\text{cm}^{-3}$ around an O star emitting 10^{48} ionizing photons per second. It is of solar metallicity, and formed by ionization of neutral gas with a standard WMN dust to gas ratio.

   a) Calculate the dust optical depth across the radius of the ionized part of the HII region at 5007Å(wavelength of the [OIII] cooling line) and at 1000Å(near the ionization edge). Interpolate an $R = 3.1$ extinction curve from those given in Fig 5.7 (p 147) of Tielens, and use the typical dust extinction to gas given in equation 5.96 (page 153, similar to the eyeball estimate given in the first lecture). [With some care you may also be able to use the right panel of Fig 5.15 as a check].

   b) Explain why virtually *all* Lyman alpha photons are absorbed by dust in an HII region, while a large fraction of the ionizing photons are able to penetrate to the edge of the Strömgren sphere, despite the fact that the dust opacity is larger for ionizing photons than for Lyman alpha.

3. **Photoevaporation of dense clumps** [40 points]

   In a traditional HII region, one considers the fate of a huge region of uniform density gas surrounding a hot star. In reality, the gas around stars is far from uniform, and there are many small dense blobs of gas. Ionization of these by nearby stars heats and pressurizes the gas on the outside of the blob, driving an outflow of ionized gas from the dense neutral blob. This photoevaporation is an important mechanism of mass transfer between media, and has major implications for star and planet formation. Many of the most popular Hubble photos are of such flows, e.g. http://hubblesite.org/gallery/album/entire/pr2009025i/large_web/ and the famous proplyds (flows from protoplanetary disks) in Orion http://hubblesite.org/gallery/album/entire/pr1994024b/. [1] In this problem you will make a quantitative model of a photoevaporative outflow, assuming it is isothermal, in photoionisation equilibrium, and in steady state (the correctness all of these simplifying assumptions need to be checked in particular applications of course).

[1] Photoevaporation has also been proposed to limit accretion rates in quasars and X-ray binaries, and limit the growth of massive stars.

a) Consider one of the proplyds in Orion, which at a distance of 300 pc subtend about 0.3 arcsec radius. Estimate the dynamical (outflow) time in terms of the Mach number $\mathcal{M}$ of the outflow, and show that the recombination time will be less than the outflow time if the hydrogen number density $n > 300\mathcal{M}^{-1}\mathrm{cm}^{-3}$.

b) Starting at about the point where the gas becomes half-ionized by the incident ionizing radiation, the increased pressure drives an outflowing photoionized wind. As discussed in class, the temperature of this wind will be nearly constant (roughly 10^4K) due to photoionization heating. Thus, unlike adiabatic winds (which asymptote to a constant velocity roughly equal to the initial sound speed), a photoionized isothermal wind keeps accelerating as it expands out. In steady state spherical symmetry, the equation of mass conservation is

$$4\pi \rho v_r r^2 = \mathrm{const} \tag{1}$$

and the equation of momentum conservation is

$$\rho v_r \frac{dv_r}{dr} + \frac{kT}{m}\frac{d\rho}{dr} = 0 \tag{2}$$

where m is the mean mass per particle (note that $(kT/m)\nabla\rho = \nabla p$, and $kT/m = c_i^2$, the square of the isothermal sound speed in the ionized gas). Defining the Mach number $\mathcal{M} = v_4/c_i$, show that these equations have the implicit solution

$$\frac{r}{r_c} = \mathcal{M}^{-1/2} \exp\left[\frac{\mathcal{M}^2 - 1}{4}\right], \tag{3}$$

$$\frac{\rho}{\rho_c} = \exp\left[-\frac{\mathcal{M}^2 - 1}{2}\right], \tag{4}$$

where the subscript c's denote the (sonic) point where the outflow reaches Mach 1, which as we already discussed must be at about the point where the flow becomes half-ionized (details require considering ionization and flow simultaneously, beyond the scope of a short homework problem. The required D-critical flow is discussed in Chapter 12 of Tielens). Check that $r = 1.11r_c$ has $\rho = 0.5\rho_c$ and $r = 1.39r_c$ has $\rho = 0.1\rho_c$ and a recombination rate only about one hundredth the rate at r_c, so the accelerating flow rapidly becomes very tenuous and highly ionized.

c) In the subsonic region just inside $r = r_c$, the cool neutral gas (which we will denote by subscript 0) must be close to pressure equilibrium with the ionization-heated gas (denoted by subscript i) on the other side of the ionization front. Show that this requires that $\rho_0/\rho_i \approx c_i^2/c_0^2$. [2]

d) For photoionization equilibrium, the number of recombinations per unit volume at each point in the wind must balance the number of new photoionizations. The number of photoionizations will be set by the incident ionizing radiation field (which in this problem we will assume to be isotropic -e.g. from a surrounding cluster of stars like Orion where $S_\infty/(2\pi) \sim 10^{11}\mathrm{cm}^{-2}\mathrm{s}^{-1}\mathrm{sr}^{-1}$ ionizing photons). Since as you showed in part b, the density drops very rapidly on a scale less than r_c, we can approximate the parts of the flow where most of the photoelectric absorption occurs as plane parallel, and calculate the solid-angle integrated flux of ionizing photons at radius r by

$$S(r) = (S_\infty/2\pi)\int_0^{\pi/2} \exp[-\tau(r)/\cos(\theta)] 2\pi \sin\theta d\theta \tag{5}$$

[2] The exact result for a flow satisfying the Jouguet condition across the front is $\rho_0/\rho_i = 2c_i^2/c_0^2$, but you don't need to do all the work required to get that factor of two.

Show that for large τ, only small θ contributes to the integral, and

$$S(r) \simeq S_\infty \frac{\exp[-\tau(r)]}{\tau(r)} \; . \tag{6}$$

e) Show that the equation of photoionization equilibrium can be written as

$$S(r)\frac{d\tau(r)}{dr} = -\alpha(T) n_e(r)^2 \tag{7}$$

where $\alpha(T)$ is the recombination coefficient and $n_e \simeq n_H$ since the gas is highly ionized in the outflow.

f) Show, by your choice of approximate evaluation of integrals or exact numerical integration, that for the flow to be self-consistent, the density at the sonic point must be

$$n_c = C\left(\frac{S_\infty}{\alpha(T) r_c}\right)^{1/2} \; , \tag{8}$$

where C is a number of order unity.

g) Show, using part 3f, that for the scale of the Orion Proplyds, the size scale you found in part 3a, and the ionizing flux given in part 3d

$$n_c \simeq 2 \times 10^4 \mathrm{cm}^{-3} \tag{9}$$

and therefore, using part 3c, that the neutral gas cloud (at temperature of $\sim$ 100K) must have density

$$n_0 \simeq 4 \times 10^6 \mathrm{cm}^{-3} \tag{10}$$

at the radius where it is being evaporated. Check using part 3a that it was OK to assume photoionization equilibrium in deriving this result.

h) What would happen if the Proplyd gas cloud were much denser or less dense than the critical value n_0 you found in part 3g?

4. **Toy Photoionization Model**

This problem will lead you through the construction of a toy photoionization model, which will allow you to explore several aspects of the physics of photoionized regions in a semi-quantitative way. We consider a gas cloud illuminated by an ionizing flux with $F(\epsilon)\, d\epsilon =$ erg cm^{-2}s^{-1} passing through the cloud in photons of energy between ϵ and $\epsilon + d\epsilon$. We take

$$F(\epsilon) = \begin{cases} 0 & \epsilon < \chi_H \\ F_H(\epsilon/\chi_H)^{-s} & \chi_H < \epsilon \end{cases} \tag{11}$$

Suppose that the gas cloud is optically thin both to the incident ionizing photons, and to all photons emitted by atoms in the cloud (i.e., a one-zone model, ignoring all radiative transfer!). The cloud is in *steady state.*

We populate our cloud with 3 types of atoms, "H," "B," and "C."

"H" is a toy hydrogen which can be ionized ($\chi_H = 13.6 = k(1.58 \times 10^5$K, but has no internal energy levels (why might this be a good approximation?). Take the photoionization cross section to be that for hydrogen

$$\sigma_H(\epsilon) \simeq \sigma(\epsilon/\chi_H)^{-2.7} \qquad \epsilon > \chi_H \tag{12}$$

$$\sigma \simeq 6 \times 10^{-18} \mathrm{cm}^2 \; . \tag{13}$$

The recombination rate is (per unit volume)

$$\alpha_H n_e n(H^+) \ , \tag{14}$$

where $\alpha_H = 3 \times 10^{-13}\mathrm{cm}^3\mathrm{s}^{-1}\mathrm{T}_4^{-1/2}$. $T_e = 10^4\mathrm{T}_4\mathrm{K}$ is the temperature of the free electron population (recall from lecture that electrons thermalize much faster than they recombine).

When an electron recombines, it loses $\sim kT_e$ of energy, plus atomic binding energy. Let us agree to count *only* energy in the free electron population, with the zero level set at the $n \to \infty$ levels of hydrogen. Then the cooling rate per unit volume due to recombination and bremsstrahlung is approximately

$$\simeq \alpha_H n_e n(H^+) kT_e \ . \tag{15}$$

a) Show that with the zero of energy defined as above, the rate of heating the free electron bath by photoionization is

$$n(H^0) \int_{\chi_H}^{\infty} (\epsilon - \chi_H)\sigma_H(\epsilon) \frac{F(\epsilon)}{\epsilon} \, d\epsilon \ , \tag{16}$$

and evaluate this in terms of the parameters given above.

b) Write the equation expressing photoionization equilibrium in terms of quantities defined above.

c) The second type of atom in our cloud, "B," is a toy atom with two levels, joined by a forbidden radiative transition. The parameters are those of the important [OIII] $\lambda 5007, 4959$ doublet: $E_{12}(B) = 2.5\mathrm{eV} = k(2.9 \times 10^4\mathrm{K})$.

Since it is a forbidden line, the Einstein A coefficient (spontaneous radiative transition rate) is small

$$A_{21}(B) = 0.01\mathrm{s}^{-1} \ . \tag{17}$$

We take the statistical weights of both levels to be $g_1 = g_2 = 1$, and the collision strength

$$\Omega_{12} \simeq 1 \ . \tag{18}$$

The B atoms are excited to level 2 by collisions with electrons, at a rate

$$C_{12}(B) = \gamma n_e e^{-E_{12}(B)/kT_e} \ , \qquad \gamma \simeq 10^{-7}\mathrm{cm}^3\mathrm{s}^{-1}\mathrm{T}_4^{-1/2} \ . \tag{19}$$

Collisions with electrons also de-excite them, at rate

$$C_{21}(B) = \gamma n_e \tag{20}$$

Write the equation expressing equilibrium between the levels 1,2 in "B" atoms.

d) The third and last type of atom in our cloud, "C" is also a two-level atom, but this time joined by a permitted (electric dipole) radiative transition. We'll take the parameters to be those of the important CIV λ 1549 transition: $E_{12}(C) = 8\mathrm{eV} = k(9.3 \times 10^4\mathrm{K})$, but could equally be approximations to any of the other strong permitted lines in hydrogen, or nitrogen, oxygen.

$$A_{21}(C) = 10^8\mathrm{s}^{-1} \tag{21}$$

$$\frac{\Omega_{12}(C)}{g_j} = 1 \ , \qquad g_1 = g_2 = 1 \tag{22}$$

Like "B," the levels of "C" are excited and de-excited by collisions with free electrons. The rates are as given for "B," but with $E_{12}(C)$ replacing $E_{12}(B)$.

Write the equation expressing equilibrium between the levels 1,2 in "C" atoms.

e) The final equation which determines the state of the cloud is the equation cooling = heating for the bath of free electrons. Write down the rates of photoionization heating, recombination cooling, and cooling by radiative de-excitation of "B" and "C" atoms.

f) You now have 4 equations describing the equilibrium state of the cloud.

Define the dimensionless variables

$$x_H = \frac{n(H^+)}{n(H)} = \frac{n_e}{n(H)} \quad , \quad x_B = \frac{n_2(B)}{n(B)} \quad , \quad x_C = \frac{n_2(C)}{n(C)} \tag{23}$$

and

$$\nu_B = \frac{n(B)}{n(H)} \quad , \quad \nu_C = \frac{n(C)}{n(H)} \ . \tag{24}$$

The x's give the fractions in excited states. The ν's give the fractional abundances of "B" and "C" relative to H.

Define the hydrogen total pressure (almost exactly the total pressure, since $\nu_B \ll 1$, $\nu_C \ll 1$)

$$p = n(H)(1 + x_H)kT_e \qquad \text{(why?)} \tag{25}$$

and the dimensionless ionization parameter

$$\Xi = \frac{\int_{\chi_H}^{\infty} F(\epsilon)\, d\epsilon}{cp} = \frac{F_H \chi_H}{(s-1)cp} \ . \tag{26}$$

Rewrite your 4 equations in terms of these dimensionless variables, Ξ, the dimensional p (or $n(H)$) and the physical constants of the problem.

g) Now solve these equations in turn: Show that the equation you found in (b) gives

$$x_H = \left[1 + \frac{\alpha_H}{c\sigma} \frac{(2.7+s)}{(s-1)} \frac{\chi_H}{kT_e} \frac{1}{\Xi}\right]^{-1/2} \ . \tag{27}$$

h) Solve your equations of (c) and (d) to give

$$x_B \;=\; \text{expression in} \quad T_e, x_H, n(H), \text{ and physical constants} \tag{28}$$

$$x_C \;=\; \text{expression in} \quad T_e, x_H, n(H), \text{ and physical constants} \tag{29}$$

where one should substitute $n(H) = \frac{p}{(1+x_H)kT_e}$, because one can't specify the density arbitrarily, but one *can* specify the pressure in astrophysical systems (why?).

i) If the expressions you obtained in (g) and (h) are substituted in the heating=cooling thermal equilibrium equation, show that the resulting equation is a nonlinear equation for T_e, depending only upon physical input parameters (Ξ, p, ν_B, ν_C) and physical constants. The solutions to this equation give the equilibrium temperatures of the cloud. It is most useful to write this equation as $\mathcal{C}(T_e) - \mathcal{H}(T_e) = 0$ where $\mathcal{C}(T_e)$ is the cooling rate *per baryon* $\left(= \frac{\text{rate/vol}}{n(H)}\right)$ and $\mathcal{H}(T_e)$ is the heating rate *per baryon* $\left(= \frac{\text{rate/vol}}{n(H)}\right)$.

Show that

$$\mathcal{C}(T_e) = \alpha_H x_H^2 n(H) k T_e + x_B \nu_B A_{21}(B) E_{21}(B) + x_C \nu_C A_{21}(C) E_{21}(C) \tag{30}$$

and

$$\mathcal{H}(T_e) = \frac{\alpha_H x_H^2 \chi_H n(H)}{(2.7 + s - 1)} \tag{31}$$

j) Write a computer program to solve this equation (graphically or otherwise). Be sure your output allows you to compare the relative importance of the various heating and cooling terms.

k) Take $s = 1.5$, $\nu_B = 10^{-4}$, $\nu_C = 10^{-4}$ (appropriate for the heavy elements which they model). For $\Xi = 1$, explore the equilibria in the range of pressures $10^4 \mathrm{cm}^{-3} {}^\circ\mathrm{K} < p/k < 10^{15} \mathrm{cm}^{-3}\ {}^\circ\mathrm{K}$. Evaluate the relative importance of cooling by H, B, and C. Can you explain why B dominates the cooling at low densities, while C dominates at high? Give an expression in terms of physical constants for the approximate density (or pressure) where the crossover in importance of B and C occurs.

l) Now take $s = 1.5$, $\nu_B = 10^{-4}$, $\nu_C = 10^{-4}$ again, but fix $p/k = 10^5 \mathrm{cm}^{-3}\mathrm{K}$, and explore the equilibria in the range $10^{-5} < \Xi < 10^5$. Plot T as a function of Ξ. Look carefully at $10^3\ {}^\circ\mathrm{K} < T < 10^7\ {}^\circ\mathrm{K}$. What heating=cooling balance is responsible for each equilibrium? Which equilibria are stable? (recall that an equilibrium is thermally stable only if

$$\frac{d}{dT} (\mathcal{C} - \mathcal{H}) \|_{\text{fixed pressure}} > 0 \qquad \text{—why?)}\ . \tag{32}$$

m) Explore additional regions of parameter space, including the limits $\nu_B, \nu_C \to 0$. Are you surprised at how important the trace elements are in thermostating the clouds?