Week 2: Classical Cosmological Tests

Cosmology, Ay127, Spring 2005

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1 Classical cosmological tests

In the following, we will assume that the Universe consists of nonrelativistic matter and possibly a cosmological constant. Before proceeding, let us justify our neglect of radiation. A variety of measurements (including, for example, simply weighing galaxies and then measuring their space density) indicate a nonrelativistic-matter density $\rho_m \simeq 0.3\rho_c \simeq 1.5 \times 10^{-6}$ GeV/cm^3 (for $h \simeq 0.7$). We observe a cosmic microwave background, a blackbody spectrum of photons, with a temperature of $T_0 \simeq 2.7$ K. From the Stefan-Boltzmann equation, this corresponds to an energy density $\rho_\gamma \simeq 3\times 10^{-10}$ GeV/cm^3. We also have good reason to believe that there is a cosmic neutrino background with an energy density roughly 2/3 that of the photon density. Thus, radiation (neutrinos plus photons) has an energy density only $\sim 3 \times 10^{-4}$ of the matter density today. Since $\rho_{\rm rad} \propto (1+z)^4$ while $\rho_m \propto (1+z)^3$, radiation will be negligible compared with matter as long as we are at redshifts less than $z_{\rm eq} = (\rho_m/\rho_{\rm rad})_0 \simeq 3000$. At earlier times, the Universe was radiation dominated.

2 Angular-diameter distance

Heuristically, if the size of an object is known, its distance can be inferred by determining how big it appears to be—i.e., the angle it subtends when we view it. In cosmology, the *angular-size distance* takes into account the effects of expansion and geometry to relate the observed angular size of an object of known proper size to its distance. This is done as follows: The coordinate distance from light emitted at time t_e to t_0 is

$$\chi = \int_{t_e}^{t_0} \frac{dt}{a} = \int_{a_e}^{a_0} \frac{da}{a\dot{a}} = \frac{1}{H_0}\int_{a_e}^{a_0} \frac{da}{a^2 E(z)} = \frac{1}{a_0 H_0}\int_0^{z_e} \frac{dz}{E(z)}, \tag{1}$$

where the function $E(z)$ describes the time evolution of the expansion rate, $H(z) = H_0 E(z)$. From the form of the metric, we know that at time t_e, the circumference of a great circle of coordinate radius χ is $2\pi a(t_e) S_\chi$, where $S_\chi = (\sinh\chi, \chi, \sin\chi)$ for open, flat, and closed universes, respectively. Therefore, if we see today an object of proper size D, then the angle it subtends on the sky is

$\theta = D/[a(t_e)S_\chi] \equiv D/d_A$, where $d_A(z_e) = a(t_e)S_\chi$. Thus,

$$d_A(z) = \frac{a_0}{1+z} \left\{ \begin{array}{c} \sinh\left[\frac{1}{a_0 H_0}\int_0^z \frac{dz}{E(z)}\right] \\ \frac{1}{a_0 H_0}\int_0^z \frac{dz}{E(z)} \\ \sin\left[\frac{1}{a_0 H_0}\int_0^z \frac{dz}{E(z)}\right] \end{array} \right\}, \tag{2}$$

for (from top to bottom) open, flat, and closed universes. For example, in an Einstein-de Sitter universe,

$$d_A(z) = \frac{2H_0^{-1}}{1+z}\left[1 - \frac{1}{\sqrt{1+z}}\right], \tag{3}$$

which, for $z \ll 1$ becomes $d_A \simeq H_0^{-1} z$, indicating that we recover the expected behavior at small distances. You can also show that this linear relation is recovered for any Ω_m or Ω_Λ. It can also be shown that to quadratic order in z, $H_0 d_a(z) \simeq z - (1/2)(3+q_0)z^2 + \cdots$, where q_0 is the deceleration parameter. For future use, it will be convenient to define a scaled distance, $y(z) \equiv H_0(1+z)d_A(z)$. Expressions for $y(z)$ involve more complicated integrals for $\Omega_m \neq 1$ and for $\Omega_\Lambda \neq 0$. For $\Omega_\Lambda = 0$ and $z \gg \Omega_m^{-1}$, $y(z) \simeq 2/\Omega_m$. In practice, it is difficult to find objects (like galaxies) of fixed known size D, making the determination of the angular-diameter distance difficult.

3 Luminosity distance

If we know the intrinsic luminosity L of a source, then we can determine its distance by measuring the energy flux F we observe from this source. The *luminosity distance* of a cosmological source is defined by $d_L^2 \equiv L/(4\pi F)$. The flux F we observe at time t_0 from a source at a distance χ is

$$F = \frac{L}{4\pi a^2(t_0) S_\chi^2 (1+z)^2}. \tag{4}$$

This result is arrived at in the following way: If the detector area is dA, the fraction of the 2-sphere, centered on the source, that is covered by the detector is $dA/[4\pi a^2(t_0)S_\chi^2]$. Then there is an additional factor of $1+z$ that is due to the redshift of photon energy, and there is another factor $1+z$ due to the redshift of the emission rate (if the source emits in its rest frame a signal with a period P, it is observed with period $(a_0/a_e)P$. Therefore, $d_L = d_A(1+z)^2$. Note again that to quadratic order, $H_0 d_L(z) = z + (1/2)(1-q_0)z^2 + \cdots$. If there is a "standard candle", a class of objects of fixed known luminosity, then the parameters H_0 and q_0 can be determined by measuring the observed flux as a function of redshift. Thus, $q_0 = \Omega_m/2 - \Omega_\Lambda$ has been obtained by measuring the quadratic correction to the Hubble law with distant (Type Ia) supernovae leading to the value $q_0 \simeq -0.55$ mentioned before.

4 Proper displacement

For a number of applications, it is important to know the proper-distance interval dl covered in a redshift interval dz. This is obtained by noting that a light ray covers a distance

$$dl = dt = \frac{da}{\dot a} = \frac{dz}{1+z}\frac{a}{\dot a}, \tag{5}$$

from which it follows that

$$\frac{dl}{dz} = \frac{H_0^{-1}}{(1+z)E(z)}. \tag{6}$$

5 The resolution of Olber's paradox

In a homogeneous, static, and infinite Universe, every line of sight eventually ends up on a galaxy, and if so, then the night sky should be bright. We can now understand why in an expanding Universe the night sky is dark. Consider a population of objects of cross section σ (e.g., $\sim \pi(10\,\mathrm{kpc})^2$ for spiral galaxies) with a constant number per comoving volume and current number density n_0. That means that the proper number density as a function of redshift is $n(z) = n_0(1+z)^3$. The probability that a given line of sight intersects such an object between z and $z + dz$ is

$$\frac{dP}{dz} = \sigma n(z)\frac{dl}{dz} = \sigma n_0 H_0^{-1}\frac{(1+z)^2}{E(z)}. \tag{7}$$

At $z \gg \Omega_m^{-1}$, $E(z) \to \Omega_m^{1/2}(1+z)^{3/2}$, and the optical depth for intersecting a galaxy out to a redshift z is

$$\tau(z) = \int^z dP = \frac{2}{3}\frac{\sigma n_0 c H_0^{-1}}{\Omega_m^{1/2}}(1+z)^{3/2}. \tag{8}$$

Ordinary galaxies have a local number density $n_g \sim 0.02\,h^3\,\mathrm{Mpc}^{-3}$, and radii $r_g \sim 10\,h^{-1}$ kpc, from which we obtain $\tau \sim 0.01\,(1+z)^{3/2}\Omega_m^{-1/2}$. Therefore, out to $z = 1$, about $0.04\,\Omega_m^{-1/2}$ of the sky is covered by galaxies, and full coverage ($\tau = 1$) is reached only at $z \sim 20\,\Omega_m^{1/3}$. Thus, Olber's paradox is explained if galaxies don't form or light up fully until $z \sim$few.

6 Number counts

We can also calculate the number of objects seen in a given redshift interval dz in a solid angle $\delta\Omega$ on the sky, under the assumption that the comoving number density of such objects remains constant. The area subtended by an angle $\delta\Omega$ at a redshift z is $\delta A = a^2 S_\chi^2 \delta\Omega$. Then, $(dl/dz)dz$ is the proper linear depth from z to $z + dz$, so the differential volume in the redshift interval dz and solid angle $\delta\Omega$ is

$$\delta V = \frac{H_0^{-1}\delta z}{(1+z)E(z)}\frac{(a_0 S_\chi)^2\delta\Omega}{(1+z)^2}. \tag{9}$$

Using $n(z) = n_0(1+z)^3$, we find that the number of galaxies in dz per steradian on the sky is

$$\frac{d\mathcal{N}}{dz} = n_0 H_0^{-3} F_n(z), \tag{10}$$

where

$$F_n(z) = \frac{[H_0 a_0 S_\chi(z)]^2}{E(z)}. \tag{11}$$

Since S_χ and $E(z)$ depend on the matter content and the geometry, measuring the number counts as a function of redshift can in principle be used to determine cosmological parameters. In practice, though, the abundances of the target populations (e.g., galaxies or clusters of galaxies) undergo evolution in complicated ways, and this evolution is difficult to disentangle from the cosmological effects.

7 "Superluminal" proper motions

The black holes that power active galactic nuclei can often emit jets with relativistic velocities. Suppose that such a source emits a jet with velocity v at an angle θ from the line of sight. Then, after a time δt, the jet will have propagated a distance $v\delta t\cos\theta$ toward us and a transverse distance $v\delta t\sin\theta$. Suppose now that the observer-source distance is D_{os}. The observer will see the signal at a transverse distance $\delta l_\perp = v\delta t\sin\theta$ after a time

$$\begin{aligned} \Delta t_{\rm obs}(\delta t) &= \delta t + \left[(D_{os} - v\delta t\cos\theta)^2 + (v\delta t\sin\theta)^2\right]^{1/2} \\ &= \delta t + \left[D_{os}^2 + (v\delta t)^2 - 2D_{os}v\delta t\cos\theta\right]^{1/2} \qquad (12) \\ &\simeq D_{os} + \delta t(1 - v\cos\theta), \qquad (13) \end{aligned}$$

Therefore, the apparent transverse velocity is

$$\frac{\delta l_\perp}{\delta t} = \frac{v\sin\theta}{1 - v\cos\theta}, \qquad (14)$$

which is maximized, for a given v, at $\cos\theta = v$. So, $(\delta l_\perp/\delta t)_{\rm max} = \gamma v$, which is faster than the speed of light for $v > c/\sqrt{2}$. If the source is at redshift z, then the observed time interval is $\delta t_o = (1+z)\delta t$, and

$$\delta\theta = \frac{\delta l_\perp}{a(z)S_\chi(z)} = \frac{\delta l_\perp(1+z)}{a_0 S_\chi(z)}. \qquad (15)$$

Therefore, the observed angular proper motion is

$$\mu = \frac{d\theta}{dt} = \frac{1}{a_0 S_\chi(z)}\frac{v\sin\theta}{1 - v\cos\theta}, \qquad (16)$$

and

$$\mu_{\rm max}(z) = \frac{\gamma v}{a_0 S_\chi(z)}. \qquad (17)$$

Since Ω_m and Ω_Λ fix $H_0 a_0 S_\chi(z)$, the maximum proper motion is determined by $H_0\gamma v$. Just for reference, observations indicate $h\gamma \simeq 10$. The idea is then to measure v, θ, and μ (assumed to be less than $\mu_{\rm max}$) to constrain Ω_m and Ω_Λ.

8 Thermodynamics in the Expanding Universe

As discovered by Penzias and Wilson in 1965, and determined much more precisely in the early 1990s by the *Far Infrared Absolute Spectrometer (FIRAS)* on NASA's Cosmic Background Explorer (COBE) satellite, the Universe is filled with a gas of microwave photons with a blackbody spectrum and a temperature $T_0 = 2.7$ K, or $T = 2.4 \times 10^{-13}$ GeV in units in which the Boltzmann constant $k_B = 1$. In other words, in every direction we look, we see a specific intensity (ergs/cm^2/sec/steradian/Hz),

$$I_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT_0} - 1}, \tag{18}$$

as a function of frequency ν. It is important to keep in mind that although this is the energy distribution that a gas of massless particles has in thermal equilibrium, the photons in this gas are most certainly *not* in thermal equilibrium. Thermal equilibrium implies that the energy-momentum distribution of the particles in the thermal bath is maintained by frequent collisions. These photons have extremely long mean-free paths through the Universe, comparable to the size of the observable Universe, and thus are not in thermal equilibrium, even though they have the energy distribution characteristic of thermal equilibrium. We refer to this gas of photons as the *cosmic microwave background (CMB).*

Interestingly enough, since the frequency of each photon scales as $1+z$ with redshift z—$\nu(z) = (1+z)\nu_0$—the frequency spectrum of the CMB always maintains a blackbody distribution, albeit one with a temperature $T(z) = T_0(1+z)$—i.e., the earlier Universe was hotter. Although they have not yet been observed, we will also see that the Universe also constains a gas of neutrinos (all three mass eigenstates) at a temperature $T_\nu = 1.96$ K. We also measure, through a variety of techniques that we will discuss in a bit, a baryon (i.e., neutrons and protons) density (in units of critial), $\Omega_b = \rho_b/\rho_c \simeq 0.019\, h^{-2}$. (And keep in mind that $\rho_c = 10^{-5}\, h^2$ GeV cm^{-3} $= 1.9\, h^2 \times 10^{-29}$ g cm^{-3}.)

Needless to say, although the Universe is quite cool and diffuse today, it must have been hotter and denser at earlier times. For example, at a redshift $z = 10^9$, the density of the Universe approached that of water, and the temperature somewhere around an MeV. It is thus reasonable to surmise that at some sufficiently early time, the contents of the Universe must have been described by a gas of elementary particles in thermal equilibrium, rather than a distribution of far-flung galaxies that interact only gravitationally. We therefore must recall some thermodynamics to describe the early Universe.

9 Review of relevant statistical mechanics and thermodynamics

In the following, we will use particle-physics units ($\hbar = c = 1 = k_B = 1$). Then a dilute gas of weakly interacting particles has a number density,

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p})\, d^3p, \tag{19}$$

an energy density,

$$\rho = \frac{g}{(2\pi)^3} \int f(\vec{p})\, E(\vec{p})\, d^3p, \tag{20}$$

and pressure,

$$P = \frac{g}{(2\pi)^3} \int f(\vec{p})\, \frac{|\vec{p}|^2}{3E}\, d^3p, \tag{21}$$

where

$$f(\vec{p}) = \left[\exp\left(\frac{E-\mu}{T}\right) \pm 1\right]^{-1}, \tag{22}$$

is the distribution function for a gas of particles in thermal equilibrium, and the plus (minus) is for Fermi-Dirac (Bose-Einstein) statistics, and g is the degeneracy factor. The chemical potentials μ_i for particle species i that undergo the reactions $i + j \leftrightarrow k + l$ is $\mu_i + \mu_j = \mu_k + \mu_l$ in chemical equilibrium. The energy E of a particle of mass m and momentum p is $E(p) = (p^2 + m^2)^{1/2}$. At very high temperatures ($T \gg m$ and $T \gg \mu$),

$$\rho = \begin{cases} (\pi^2/30) g T^4 & \text{bosons} \\ (7/8)(\pi^2/30) g T^4 & \text{fermions} \end{cases}, \tag{23}$$

$$n = \begin{cases} [\zeta(3)/\pi^2] g T^3 & \text{bosons} \\ (3/4)[\zeta(3)/\pi^2] g T^3 & \text{fermions} \end{cases}, \tag{24}$$

where $\zeta(3) \simeq 1.2$ is the Riemann zeta function, and

$$P = \frac{1}{3}\rho. \tag{25}$$

For degenerate fermions ($\mu \gg T$) in the relativistic limit (this limit is important for white dwarfs (near the high-mass end) and neutrons stars, but not so much for cosmology),

$$\rho = \frac{g\mu^4}{8\pi^2}, \qquad n = \frac{g\mu^3}{6\pi^2}, \qquad P = \frac{g\mu^4}{24\pi^2}, \tag{26}$$

and in the nonrelativistic limit ($m \gg T$),

$$n = g\left(\frac{mT}{2\pi}\right)^{2/3} e^{-(m-\mu)/T}, \qquad \rho = mn, \qquad P = nT \ll \rho. \tag{27}$$

For $T \gg m$ and $T \gg \mu$, the mean particle energy is

$$\langle E \rangle \equiv \frac{\rho}{n} = \begin{cases} [\pi^4/30\zeta(3)]T \simeq 2.701\, T & \text{bosons} \\ \frac{7\pi^4}{180\zeta(3)} T \simeq 3.151\, T & \text{fermions} \end{cases} \tag{28}$$

Degenerate relativistic fermions have mean particle energy $\langle E \rangle \equiv \rho/n = (3/4)\mu$. For a nonrelativistic gas, $\langle E \rangle = m + (3/2)T \simeq m$.

10 Particle-antiparticle balance

The possibilty for, e.g., brehmsstrahlung reactions ($e^- + p \leftrightarrow e^- + p + \gamma$), implies that the photon has zero chemical potential, $\mu_\gamma = 0$ in chemical equilibrium. If so, then since a charged particle X^+ and its antiparticle X^- can annihilate to photons, $X^+X^- \leftrightarrow \gamma\gamma$, we must have $\mu_+ = -\mu_-$. If so, then the formula for n above yields for fermions a particle-antiparticle asymmetry (in terms of the chemical potential μ_+,

$$n_+ - n_- = \begin{cases} (gT^3/6\pi^2)\left[\pi^2(\mu/T) + (\mu/T)^3\right] & T \gg m, \\ 2g(mT/2\pi)^{3/2}\sinh(\mu_+/T)e^{-m/T} & T \ll m, \end{cases} \tag{29}$$

The total energy density and pressure are the sum of the contributions from each particle species in the thermal bath. If there is no particle-antiparticle asymmetry (or no significant particle-antiparticle asymmetry), then the density and pressure contributed by nonrelativistic particles is exponentially suppressed. In this case, the pressure and energy density are dominated by the relativistic particles, and we arrive at a radiation energy density,

$$\rho_R = (\pi^2/30) g_* T^4, \qquad P_R = \rho_R/3, \tag{30}$$

where

$$g_* \equiv \sum_{i=\text{bosons}} g_i (T_i/T)^4 + \frac{7}{8} \sum_{i=\text{fermions}} g_i (T_i/T)^4, \tag{31}$$

is the effective number of relativistic degrees of freedom (and the sum is taken only over relativistic species, $m_i \ll T$). For example, today, g_* receives contributions from the two degrees of freedom of the photon plus 3×2 neutrino degrees of freedom with $T_\nu = (4/11)^{1/3} T_\gamma$, so $g_*(T \ll \text{MeV}) = 2 + (7/8) 6 (4/11)^{4/3} \simeq 3.36$. For temperatures MeV$\ll T \ll 100$ MeV, $T_\nu = T_\gamma$ (as we will see), and the electron is relativistic ($m_e \ll T$), so $g_* = 2 + 6(7/8) + 4(7/8) = 10.75$. At higher temperatures, quarks contribute, as do muons, τ leptons, and at even higher temperatures, the electroweak gauge bosons $W^\pm$ and Z^0 and Higgs boson(s). For the standard electroweak model (with one Higgs doublet), $g_* = 106.75$ at $T \gg 100$ GeV.

During radiation domination ($z \gtrsim 22,000\, \Omega_m h^2$ ($t \lesssim 10^5$ yr), $\rho \simeq \rho_R$, $p \simeq \rho/3$, and the scale factor $a(t) \propto t^{1/2}$. At these times, $H = (8\pi G \rho_R/3)^{1/2} = 1.66\, g_*^{1/2} T^2/m_{\text{Pl}}$, where $m_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$ GeV is the Planck mass. Under the approximation that $g_* \simeq$constant, the age of the Universe during radiation domination is $t \simeq (2H)^{-1} \simeq 0.301\, g_*^{-1/2} m_{\text{Pl}}/T^2 \simeq (T/\text{MeV})^{-2}$ sec. Therefore, the Universe was roughly one second old when the temperature was an MeV, and it scales roughly as T^{-2}.

11 Entropy

When the expansion timescale of the Universe is long compared with the timescales for reactions that maintain thermal equilibrium, then the cosmological gas can be considered to be in thermal equilibrium, but undergoing adiabatic changes with the slow expansion. Under these conditions, the entropy per comoving volume will be constant. Returning to the second law of thermodynamics, we have

$$T dS = d(\rho V) + P dV = d[(\rho + P)V] - V dP. \tag{32}$$

We then note that if $\mu = 0$ [so that the T dependence of f appears only in the combination E/T, then it can be shown (from the expression for P in terms of $f(E)$) that $dP = (\rho + P)(dT/T)$, so

$$dS = \frac{1}{T} d[(\rho + P)V] - (\rho + P)V \frac{dT}{T^2} = d\left[\frac{(\rho + P)V}{T} + \text{constant}\right]. \tag{33}$$

Therefore, the entropy per comoving volume is $S = a^3(\rho + P)/T$. But if the expansion is adiabatic, then $dS = 0$, and so the *entropy density* $s \equiv S/V = (\rho + p)/T \propto a^{-3}$ as long as local thermodynamic equilibrium is maintained. What this means is that if the energy density is dominated by radiation, then $s \propto a^{-3}$, where $s = (2\pi^2/45) g_{*s} T^3$, and

$$g_{*s} \equiv \sum_{i=\text{bosons}} g_i (T_i/T)^3 + (7/8) \sum_{i=\text{fermions}} g_i (T_i/T)^3, \tag{34}$$

and as before, the sum is taken only over relativistic species, $m_i \ll T$. Note that g_{*s} is almost always equal to g_*, and differs only if some species becomes thermally decoupled from the rest of the plasma. It is important to note that since $s \propto a^{-3}$, the temperature T is $T \propto a^{-1}$ only as long as g_{*s} remains constant. When the temperature T drops below the mass m_i of some particle in the thermal bath, then the temperature T drops a little more slowly with the expansion than $1/a$ as g_{*s} decreases. Thus, for example, when the temperature drops below m_e, electrons and positrons annihilate to produce additional photons. It is often misstated that electron-positron "heats" the photons. This is not true. What happens is that when electrons and positrons annihilate, they transfer their entropy to the photons, and this simply slows the temperature drop relative to $1/a$. Note that $s = 1.80\, g_{*s} n_\gamma$, and today, the photon number density is $n_{\gamma 0} \simeq 411\,\mathrm{cm}^{-3}$, and $s_0 = 7.04\, n_\gamma \simeq 3000\,\mathrm{cm}^{-3}$.

Since sa^3 is constant, we can use s to mark comoving volumes, and also the relation between scale factor a and temperature T: i.e., $g_{*s}T^3a^3$ =constant. Thus, for example, if baryon number is conserved, then $n_B/s \equiv (n_b - n_{\bar{b}})/s$ is the same at all times. It is common to hear people speak of the baryon-to-photon ratio $\eta \equiv (n_B/n_\gamma 0)_0 = 1.8\, g_{*s}(n_B/s)$, but this is not constant (if g_{*s} changes). In particular, the photon number, $N_\gamma = a^3 n_\gamma$ increases by 11/4 at e^+e^- annihilation near $T \sim 0.5$ MeV, since $g_{*s} = 2 + 4(7/8)$ before and $g_{*s} = 2$ afterwards. This is also why the neutrino temperature is $(4/11)^{1/3}$ relative to that of the photons. Neutrinos decouple at a temperature $T \sim MeV$, *before* e^+e^- recombination. Thus, while the (decoupled) neutrino temperature is falling as $T_\nu \propto a^{-1}$, the photon temperature is dropping as $T_\gamma \propto g_{*s}^{-1/3} a^{-1}$.

Note finally that since the momentum p of a massive particle decreases as $p \propto a^{-1}$, while its kinetic energy is $E = p^2/2m$ a massive decoupled species maintains a thermal energy distribution, but with a temperature $T \propto a^{-2}$. And, of course, its density decreases as a^{-3}. If a massive species decouples when it is relativistic ($m_D \ll T$), it does *not* maintain a thermal energy distribution when the temperature drops to $T \lesssim m$. Thus, for example, neutrinos decouple when their temperature is $T \sim$ MeV, and their temperature today is $T_{\nu 0} \sim 2 \times 10^{-4}$ eV. If they have a mass $m_\nu \gtrsim T_{\nu 0}$, then they are moving nonrelativistically today, and their energy distribution is *not* a Fermi-Dirac distribution.