Week 6: Inflation and the Cosmic Microwave Background

Cosmology, Ay127, Spring 2008

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1 Motivation

The standard hot big-bang model with an (flat) FRW spacetime accounts correctly for the observed expansion, the CMB, BBN, etc. However, it leaves a number of questions unanswered.

The flatness problem. First of all, there is the flatness problem. I.e., why is our Universe so close to flat? One possibility is that it simply began as flat; i.e., with zero curvature. In some sense, this is strange because even if we started with a precisely flat Universe, then if there were even tiny density perturbation in the initial state, then an observable region of the Universe would have locally a density different than the critical density. Thus, imagine there was some initial nonzero curvature, $k \neq 0$ and ignore the cosmological constant since it remains dynamically negligible until a redshift $z \sim 1$. Then, at redshifts $z \gtrsim 1$, Ω_m is extremely close to 1. The Friedmann equation can be rearranged to give

$$\Omega_m - 1 = \frac{k}{a^2 H^2}. \tag{1}$$

During matter (radiation) domination, $a \propto t^{2/3}$ ($a \propto t^{1/2}$), and $H \propto 1/t$ at all times, so if Ω_m is not precisely equal to 1, then it diverges from 1 with the expansion of the Universe. For $\Omega_m \simeq 0.3$ today, the matter density at BBN must have been $|\Omega_m(t_{\rm bbn}) - 1| \lesssim 10^{-16}$, and at the time of the quantum-gravity event that presumably gave rise to the FRW Universe, $|\Omega_m(t_{\rm Pl}) - 1| \lesssim 10^{-60}$. In other words, the Universe would have had to be *extremely* close to flat in the initial state, or put another way, could have tolerated no more than the very tiniest density fluctuations, no more than 1 part in 10^{60}. Put another way, if the Universe were born at the Planck time with equal energy density in the curvature and matter degrees of freedom, it would have survived no longer than a Planck time; the problem is sometimes then phrased as "why is the Universe so old?" This is also sometimes referred to as the "Dicke coincidence," although it was noted presumably much earlier by Einstein, who therefore concluded that the Universe must be precisely flat.

Horizon problem. We know that CMB photons last scattered at a redshift $z_{lss} \simeq 1100$ when the Universe was $t_{\rm lss} \simeq 380,000$ years old, and that today it is $t_0 \simeq 13.8$ billion years old and very close to flat. We can thus infer that a causally connected region at the surface of last scatter subtends an angle $\theta \simeq (1 + z_{lss})(t_{\rm lss}/t_0) \sim 1^\circ$. However, there are 4π steradians$\simeq 40,000$ square degrees on the sky. We are therefore looking at roughly 40,000 causally disconnected patches of the early Universe when we look at the CMB. Yet each has a temperature that is the same to one part in 10^5. How did these causally disconnected regions of the early Universe know to have the same temperature? This is the horizon, causality, or smoothness problem. A related problem is that the Universe must have also been very smooth on very small scales. The horizon at the time of BBN enclosed roughly a solar mass of material, more than 20 orders of magnitude less mass than the horizon encloses today. The predicted light-element abundances are nonlinear functions of the baryon density. If there were density fluctuations of order unity on solar-mass scales at the time of BBN, then the observed light-element abundances would be different from those that are observed and that are observed to be in good agreement with the predictions. We therefore know that the Universe must have been smooth on small scales as well as large.

Monopole problem. Grand unified theories predict the existence of magnetic monopoles, topological defects with masses $\sim M_{\rm GUT} \sim 10^{15}$ GeV. According to the Kibble mechanism, roughly one such monopole is produced in every Hubble volume at the GUT phase transition near $T \sim 10^{15}$ GeV. You will show in a homework problem that this would result in a monopole density many orders of magnitude greater than the critical density.

Acausal primordial perturbations. We see in the CMB primordial density fluctuations $(\delta\rho/\bar{\rho}) \sim 10^{-5}$ with a nearly scale-invariant spectrum. We also see that they are seemingly acausal; i.e., there are Fourier modes of the perturbations that have wavelengths larger than the horizon size at the surface of last scatter. Where did these come from?

2 Inflationary dynamics

As the Universe expands [i.e., the scale factor $a(t)$ increases], the Hubble length increases. During matter and radiation domination, $\ddot{a} < 0$, and so

$$\frac{d}{dt}\left[\frac{H^{-1}}{a}\right] > 0. \tag{2}$$

That is, the Hubble distance increases more rapidly than the scale factor, and as a result, with time, an observer sees larger comoving volumes of the Universe, and objects and information enter the horizon.

If, however, $\ddot{a} < 0$, which requires an equation of state $p = w\rho$ with $w < -1/3$, then

$$\frac{d}{dt}\left[\frac{H^{-1}}{a}\right] < 0. \tag{3}$$

In this case, an observer sees with time a smaller comoving patch (even though the physical or proper size of the observable patch may still be increasing), and objects/information/perturbations exit the horizon. In this way, the Universe becomes increasingly smooth. This solves the smoothness problem. With the required equation of state, the matter term, $\rho \propto a^{-3(1+w)}$ decreases less slowly than the curvature term, $\propto 1/a^2$, in the Friedmann equation, and so the curvature gets driven to zero, even if it was nonzero initially. Also, with $w < -1/3$, the scale factor is $a \propto t^\alpha$ with $\alpha > 1$. Therefore, the particle horizon, $a_0 \int_0^{a_0} dt/a(t)$ diverges at early times, eliminating the horizon problem and enabling a mechanism to produce acausal perturbations. Finally, the abundance of monopoles gets diluted by the "superluminal" (i.e., $\ddot{a} > 0$) expansion.

To accomplish $\ddot{a} > 0$, inflation simply postulates the existence of some new scalar field $\phi(\vec{x}, t)$ with a potential-energy density $V(\phi)$. If the scalar field is homogeneous (i.e., $\vec{\nabla}\phi = 0$), then its energy density is

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{4}$$

while its pressure is

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{5}$$

The equation of motion for the scalar field is in Minkowski space, $\ddot{\phi} - \nabla^2\phi + V'(\phi) = 0$. In an FRW spacetime, $\ddot{\phi} - \nabla^2\phi + V'(\phi) = 0$, so the expansion acts as a friction term (assuming here $\vec{\nabla}\phi = 0$).

The idea of inflation is that if $V(\phi)$ is nonzero and sufficiently flat, then the friction term in the ϕ equation of motion may be large enough so that the field, if displaced from its minimum, will roll slowly down the potential. If so, it may be possible that $(1/2)\dot{\phi}^2 \ll V(\phi)$, and if so, then $p \simeq -\rho$. If, moreover, this false-vacuum energy dominates the energy density of the Universe, then the condition $(w < -1/3)$ for inflation will be satisfied.

More carefully, suppose that the energy density of the Universe is dominated by that of the scalar field. Then the Friedmann equation (the equation of motion for the scale factor) becomes

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_{\rm Pl}^2}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right], \tag{6}$$

while the scalar field has an equation of motion,

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}. \tag{7}$$

Again, if $\dot{\phi}^2 < 2V(\phi)$, then $w < -1/3$ and we get inflation. Even if this condition is not satisfied initially, it is likely to occur soon, as (for a very flat potential), $V \simeq$constant with the expansion, while $\dot{\phi}^2 \propto a^{-6}$.

Slow-roll approximation. For a given potential $V(\phi)$, the Friedmann equation and the scalar-field EOM can both be satisfied numerically, if not analytically. However, there is a "slow-roll" approximation that is almost always good, and that illustrates the physics clearly. In the slow-roll approximation, we assume $\dot{\phi}^2 \ll V(\phi)$. Then

$$H^2 \simeq \frac{8\pi V}{3m_{\rm Pl}^2}. \tag{8}$$

If so, then the V'' term in the scalar-field EOM will be negligible and the scalar-field EOM will simplify to

$$3H\dot{\phi} \simeq -V'(\phi), \tag{9}$$

if the *slow-roll conditions*,

$$\epsilon(\phi) \equiv \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1, \tag{10}$$

and

$$|\eta(\phi)| \equiv \left|\frac{m_{\rm Pl}^2}{8\pi}\frac{V''}{V}\right| \ll 1, \tag{11}$$

are satisfied.

Chaotic inflation. For example, consider the potential $V(\phi) = (1/2)m^2\phi^2$, and suppose that the initial scalar-field value is $\phi_i \gg m_{\rm Pl}$ (but still small enough that $V \lesssim m_{\rm Pl}^4$, so that our classical analysis applies). Then $V' = m^2\phi$, and $V'/V = 2/\phi < 4\sqrt{\pi}/m_{\rm Pl}$ as long as $\phi > m_{\rm Pl}/2\sqrt{\pi}$, and $V''/V = 1/2\phi^2 < 8\pi/m_{\rm Pl}^2$ for $\phi > m_{\rm Pl}/4\sqrt{\pi}$. Thus, as long as these conditions are satisfied, the field will roll slowly down the potential and inflation will occur. Then, when it rolls down to $\phi \lesssim m_{\rm Pl}/2\sqrt{\pi}$, the slow-roll approximation will break down, inflation will end, and the field will then undergo coherent oscillations about the minimum of the potential:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0, \tag{12}$$

but with $H \ll m$ now, this has solution $\phi(t) \propto \exp[\pm imt - (3/2)Ht]$, so the energy density in the scalar field, averaged over an oscillation cycle, becomes $\bar{\rho}(t) = \dot{\phi}^2 \propto e^{-3Ht}$ (not that the first equality follows from the virial theorem for a harmonic oscillator). I.e., the energy density decays like nonrelativistic matter. This makes sense, as coherent oscillations of the scalar field describe a gas of zero-momentum (i.e., cold) particles of mass m. If the scalar field has some coupling to standard-model particles, then the decay of these *inflaton* particles to ordinary particles will produce the primordial thermal bath, a process called *reheating*.

We can now justify our statement that inflation occurs if the slow-roll conditions occur. Most generally, $\ddot{a} = \dot{H} + H^2 > 0$ is always true if $\dot{H} > 0$. If for some reason $\dot{H} < 0$, then the condition $\ddot{a} > 0$ requires $-\dot{H}/H^2 < 1$, but $-\dot{H}/H^2 \simeq (m_{\rm Pl}^2/16\pi)(V'/V)^2 = \epsilon$. So $\epsilon < 1$ guarantees inflation.

3 Amount of inflation

The number of e-foldings of inflation between the end of inflation and a time t during inflation is

$$N(t) \equiv \ln\frac{a(t_{\rm end})}{a(t)} = \int_t^{t_{\rm end}} H\,dt \simeq \frac{1}{m_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi} \frac{V}{V'}\,d\phi. \tag{13}$$

4 Evolution of scales

The biggest comoving scales exit the horizon first during inflation, and they are the last to re-enter the horizon during matter or radiation domination. Let us therefore consider the number of e-foldings required to solve the horizon problem. Consider a physical wavenumber $k_{\rm phys}$. Its ratio to the Hubble scale today is

$$\frac{k_{\rm phys}}{a_0 H_0} = \frac{a_k H_k}{a_0 H_0} = \frac{a_k}{a_{\rm end}}\frac{a_{\rm end}}{a_{\rm reh}}\frac{a_{\rm reh}}{a_{\rm eq}}\frac{a_{\rm eq}}{a_0}\frac{H_k}{H_0}, \tag{14}$$

where a_k and H_k are the scale factor and Hubble parameter when this particular wavenumber exits the horizon, $a_{\rm end}$ is the scale factor at the end of inflation, and $a_{\rm eq}$ is the scale factor at matter-radiation equality. Plugging in numbers, we find that the number of e-foldings between the end of inflation and the time at which the wavenumber k exits the horizon is

$$N(k) = 62 - \ln \frac{k_{\rm phys}}{a_0 H_0} - \ln \frac{10^{16}\,{\rm GeV}}{V_k^{1/4}} + \ln \frac{V_k^{1/4}}{V_{\rm end}} - \frac{1}{3} \ln \frac{V_{\rm end}^{1/4}}{\rho_{\rm reh}^{1/4}}. \qquad (15)$$

5 An Exact Solution

There are a few models for which the scalar-field and scale-factor EOMs can be solved exactly. The first example is power-law inflation, which features a potential,

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{m_{\rm Pl}}\right). \qquad (16)$$

This model has a scale factor $a(t) = a_0 t^p$ and the scalar field moves with time t according to

$$\frac{\phi}{m_{\rm Pl}} = \sqrt{2p} \ln\left(\sqrt{\frac{V_0}{p(3p-1)}}\frac{t}{m_{\rm Pl}}\right). \qquad (17)$$

There is inflation for $p > 1$, and the slow-roll parameters are $\epsilon = \eta/2 = 1/p$, and $w = 2/3p$. In this model, there is no end to inflation.

6 Inflationary perturbations

Inflation was originally postulated in order to solve the monopole, horizon, and flatness problems. However, quite remarkably, it was very soon realized that inflation could do much more, in particular produce the primordial perturbations required for large-scale structure. The recent round of CMB experiments confirm that primordial perturbations are of the type and have the spectrum predicted by the simplest slow-roll models of inflation. This leads us to take these models even more seriously than before. Inflation moreover predicts (as Mark Wise was one of the first to show) a stochastic background of gravitational waves. A precise derivation of these results requires the full machinery of quantum field theory in curved spacetime and of relativistic cosmological perturbations, both of which are beyond the scope of this class. Still, we can try to see heuristically where these predictions for density perturbations (scalar metric perturbations) and gravitational waves (tensor metric perturbations) come from.

Vacuum Fluctuations in Inflation. In an FRW spacetime, the KG equation becomes

$$\ddot{\phi} + 3H\dot{\phi} + \nabla^2\phi + dV/d\phi = 0, \qquad (18)$$

where the dot denotes derivative wrt t, and here $\nabla^2 = (1/a^2)(\partial^2/\partial x_i^2)$; i.e., it is a physical Laplacian. In some theories, there may also be a direct coupling of the scalar field to gravity through an additional term $-\xi R\phi^2$ on the left-hand side, where ξ is a coupling constant, and $R = -(\rho - 3P)/m_{\rm Pl}^2$ is the Ricci scalar; during inflation $R = -12H^2$. If $\xi = 0$, the field is said to be "minimally coupled."

Classically, during inflation, the scalar field is homogeneous, $\phi(\vec{x},t) = \phi(t)$. However, there may still be quantum-mechanical fluctuations, as we now show. To do so, we write $\phi(\vec{x},t) = \bar{\phi}(t) + \delta\phi(\vec{x},t)$, where $\vec{x}$ is a comoving FRW coordinate. The perturbation $\delta\phi$ satisfies

$$(\ddot{\delta\phi}) + 3H(\dot{\delta\phi}) - \nabla^2\delta\phi + m^2\delta\phi = 0, \tag{19}$$

where $m^2 \equiv V''(\phi)$. In Fourier space,

$$\delta\ddot{\phi}_{\vec{k}} + 3H\delta\dot{\phi}_{\vec{k}} + \left(\frac{k}{a}\right)^2 \delta\phi_{\vec{k}} + \frac{1}{2}m^2\delta\phi_{\vec{k}} = 0. \tag{20}$$

During the inflationary epoch, ($m^2/H^2 \ll 1$ from the slow-roll condition (which guarantees that $V''/V \ll m_{\rm Pl}^2$). We are therefore left with

$$\delta\ddot{\phi}_{\vec{k}} + 3H\delta\dot{\phi}_{\vec{k}} + \left(\frac{k}{a}\right)^2 \delta\phi_{\vec{k}} = 0. \tag{21}$$

At early times during inflation, before horizon exit, $k/a \gtrsim H$, meaning the physical wavelength of the mode is smaller than the horizon, and the equation of motion is simply $\delta\ddot{\phi}_{\vec{k}} + (k/a)^2 = 0$; i.e., the usual flat-space equation, and the solutions oscillate as the do ordinarily. At later times, after horizon exit, $k/a \lesssim H$, and the equation of motion becomes $\delta\ddot{\phi}_{\vec{k}} + 3H\delta\dot{\phi}_{\vec{k}} = 0$. The solution to this equation is $\delta\phi_{\vec{k}} \rightarrow$constant.

Soon after the beginning of inflation, the energy density in radiation and matter redshifts away, and we are left with a vacuum. We now solve the equation of motion,

$$\delta\ddot{\phi}_{\vec{k}} + 3H\delta\dot{\phi}_{\vec{k}} + \left(\frac{k}{a}\right)^2 \delta\phi_{\vec{k}} = 0. \tag{22}$$

This equation has two solutions. Assuming $H \simeq$constant near horizon crossing,

$$\delta\phi_{\vec{k}}(t) = L^{-3/2}\frac{H}{(2k^3)^{1/2}}\left(i + \frac{k}{aH}\right)e^{ik/aH}. \tag{23}$$

Expanding about some time $t = T$ well before horizon exit,

$$\frac{k}{aH} = \left(\frac{k}{aH}\right)_{t=T} - \frac{k}{a}(t-T) + \cdots, \tag{24}$$

where the higher-order terms are negligible if $|t-T|^{-1} \ll H^{-1}$. In this case, the phase factor changes only slowly, and identifying $E_k = k/a$, we see that the early-time behavior of the solution is

$$\delta\phi_{\vec{k}}(t) = \left(\frac{1}{aL}\right)^{3/2}\sqrt{\frac{1}{2E_k}}e^{-iE_kt}. \tag{25}$$

This is as it should be: at early times, the physical wavelength of the perturbation is small compared with the horizon and we recover the usual Minkowski-space behavior of the mode function; i.e., it oscillates in time—*the amplitude of a Fourier mode of a scalar field behaves like a simple harmonic oscillator.* In quantum mechanics, a harmonic oscillator has a zero-point energy; it has a nonzero amplitude even in the ground state. The same is true for each Fourier amplitude of the scalar field. Therefore, even in a vacuum, the scalar field is oscillating, and the amplitude of this oscillation is fixed by the Heisenberg uncertainty principle. From quantum mechanics, the position x in a harmonic oscillator in its ground state has an expectation value $\langle x^2 \rangle \propto \hbar$. The same is true for the Fourier amplitude, $\langle |\delta\phi_k|^2 \rangle = (aL)^{-3}/(2E_k)$.

This early-time behavior, familiar to us from quantum field theory in Minkowski space, normalizes the amplitude of the zero-point oscillation. However, a time t a few epochs after horizon exit, the late-time solution of the mode function tells us that

$$\langle |\delta\phi_{\vec{k}}|^2 \rangle = \frac{H^2(t_\star)}{L^3 k^3}; \tag{26}$$

i.e., what was oscillating before horizon exit becomes frozen after horizon exit. In other words, *quantum-mechanical zero-point oscillations become classical perturbations $\delta\phi(\vec{x},t)$ to the scalar field after horizon exit.* Moreover, the power spectrum for the scalar field after horizon exit (which occurs at time t_* defined by $a_k H_k(t_*) = k$) is

$$P_\phi(k, t_\star) = V \frac{k^3}{2\pi^2} |w_k|^2 = \left[\frac{H(t_*)}{2\pi}\right]^2. \tag{27}$$

During slow-roll inflation, $H \simeq$constant, so the power spectrum for ϕ resulting from inflation is

$$P_\phi(k) = \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH}. \tag{28}$$

Moreover, since ϕ is an effectively massless scalar field when these perturbations are produced, every $\vec{k}$ mode is uncorrelated with every other, and ϕ is a Gaussian random field. Strictly speaking, there will be some nonvanishing higher-order terms in the ϕ Lagrangian, and so there will be some correlations between Fourier modes and thus some non-Gaussianity. This is in fact a subject of intense current study. However, the bottom line is that these departures from Gaussianity are expected to be small in the most generic inflationary model.

To get from scalar-field perturbations to primordial density perturbations precisely again requires relativistic cosmological perturbations beyond the scope of this class. However, heuristically, if there are scalar-field perturbations $\delta\phi$, then the energy density of the Universe, $V(\phi)$ during inflation, will receive perturbations $\delta V \simeq (\partial V/\partial\phi)\delta\phi$. The result of the nasty calculations, which we will have to just take on faith at this point, is that the power spectrum for the primordial gravitational-potential perturbation will be,

$$\Delta\Phi^2(k) = \frac{128\pi V^3}{m_{\rm Pl}^6 (V')^2} = \frac{8}{3\pi^2} \frac{V}{m_{\rm Pl}^2 \epsilon}, \tag{29}$$

where V and ϵ are evaluated at $k = aH$. (**Warning:** the prefactors in this equation may be off by factors of order unity.)

The Differential Microwave Radiometer (DMR) experiment aboard NASA's Cosmic Background Explorer (COBE; 1991–4) discovered temperature fluctuations at large angular scales, or large distance scales, in the CMB. Roughly speaking, what they found was temperature fluctuations corresponding to

$$\left[\Delta\Phi^2(k)\right]^{1/2} = 5 \times 10^{-5}, \tag{30}$$

at $k = a_0 H_0$, from which we infer that $V^{1/4}/\epsilon^{1/4} = 0.027\, m_{\rm Pl}/8\pi = 6.6 \times 10^{16}$ GeV. Since $\epsilon \lesssim 1$, we infer that $V^{1/4} \lesssim 6 \times 10^{16}$ GeV, assuming, of course, that all of the temperature fluctuation is due to primordial density perturbations. The bound is even stronger if there is some other source of $\Delta T/T$.

Spectral index for primordial density perturbations. We define the spectral index for the matter power spectrum to be the logarithmic derivative of the power with respect to wavenumber,

$$n_s(k) - 1 \equiv \frac{d \ln P_{\mathcal{R}}}{d \ln k}. \tag{31}$$

The scale factor a varies much more rapidly than H during inflation, and we evaluate the power-spectrum amplitude at $k = aH$. Therefore,

$$d \ln k = \frac{dk}{k} = \frac{H da}{Ha} = \frac{da}{a} = \frac{\dot{a}}{a} dt = H\,dt, \tag{32}$$

but $dt = -(3H/V')d\phi$, so

$$\frac{d}{d \ln k} = -\frac{m_{\mathrm{Pl}}^2}{8\pi} \frac{V'}{V} \frac{d}{d\phi}. \tag{33}$$

A bit of algebra gives us

$$\frac{d\epsilon}{d \ln k} = 2\epsilon\eta - 4\epsilon^2, \tag{34}$$

and $n_s - 1 = -6\epsilon + 2\eta$. Observations suggests (at 3σ) $|n - 1| \lesssim 0.3$. Note also that the power-law index "runs" with scale; i.e., the primordial power spectrum is not a pure power law. In particular,

$$\frac{dn}{d \ln k} = -16\epsilon\eta + 24\epsilon^2. \tag{35}$$

Note that we expect $n \simeq 1$, the "Peebles-Harrison-Zelddovich" spectrum. Note that $n > 1$ ($n < 1$) is referred to as a blue (red) spectrum. There have been lots of attempts to measure the running of the spectral index by comparing power on CMB scales ($\simeq 10^4$ Mpc) with power measured at galaxy-survey scales ($\simeq 10$ Mpc). The results suggest that the primordial power spectrum is quite close to a pure power law.

7 Gravitational waves (tensor metric perturbations)

Just as Maxwell's equations allow for electromagnetic waves, propagating disturbances in the electromagnetic fields, Einstein's equation allows for *gravitational waves*, propagating disturbances in the gravitational field. The gravitational-wave amplitude h satisfies (in a vacuum) a wave equation (in an expanding Universe), just as the amplitude of an electromagnetic field does. In Fourier space, the amplitude $h_{\vec{k}}$ of the $\vec{k}$ Fourier mode satisfies,

$$\ddot{h}_{\vec{k}} + 3H\dot{h}_{\vec{k}} + (k/a)^2 h_{\vec{k}} = 0, \tag{36}$$

where the dot denotes derivative with respect to time t. This is exactly the same equation as that satisfied by the massless scalar field, and we know that modes of the scalar field are excited during inflation. The same will also happen for the gravitational field. We thus expect inflation to produce a stochastic gravitational-wave background with power spectrum,

$$P_{\mathrm{GW}}(k) \propto \left(\frac{H}{2\pi}\right)^2. \tag{37}$$

Of course, we still need to get the constant of proportionality right in the final step, particularly since the power-spectrum amplitude for the amplitude h should be dimensionless. It stands to reason that the constant should be $\propto m_{\rm Pl}^{-2}$ (what else could it be?). Again, with a bit of quantum field theory and general relativity, the correct result turns out to be

$$P_{\rm GW}(k) = \frac{16\pi}{m_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2\bigg|_{k=aH} = 2P_+(k) = 2P_\times(k). \tag{38}$$

Note that like electromagnetic waves, gravitational waves come in two different polarization states, the $+$ and $\times$ polarization states.

It is important to note that this is the *primordial* power spectrum, the amplitude of the gravitational wave when it exits the horizon during inflation. As for the scalar field, the solution to the mode function is roughly constant while the mode is outside the horizon, and so this is the power spectrum for the gravitational-wave amplitude when it re-enters the horizon during RD or MD. Once the mode enters the horizon, the solution to the mode function is $h_k \propto a^{-1}$, and so the power spectrum today is suppressed relative to the primordial power spectrum by a factor $[a(k)/a_0]^2$, where $a(k)$ is the scale factor at $k = aH$ during MD or RD. During MD, $aH \propto a^{-1/2}$ and $a(k) \propto k^{-2}$ at horizon crossing. Therefore, the power-spectrum for wavenumbers $k < k_{\rm eq}$ (i.e., the long-wavelength modes that enter the horizon during matter domination) will be $P_h(k) \propto k^{-4}H^2$, or $h_k \propto H/k^2$. Likewise, during RD, $aH \propto a^{-1}$, so $a(k) \propto k^{-1}$, and so for $k > k_{\rm eq}$ (the shorter-wavelength modes that enter the horizon during radiation domination) $P_h(k) \propto H^2/k^2$, or $h_k \propto H/k$.

Gravitational waves are ripples in spacetime, and we know that in general relativity, the propagation of photons is affected by the curvature of spacetime. CMB photons propagating through an FRW spacetime with a stochastic gravitational-wave background will therefore develop temperature fluctuations. COBE saw temperature fluctuations with an rms amplitude $\Delta T/T \simeq 10^{-5}$ on the largest angular scales. Roughly speaking, we expect a gravitational wave of amplitude h to induce a temperature fluctuation $\Delta T/T \sim h$. The largest angular scales correspond to wavelengths comparable to the horizon, and so we expect for these wavelengths, which are just entering the horizon, $h \sim (\sqrt{8\pi}/m_{\rm Pl})(H/2\pi)$, where $H \simeq \sqrt{8\pi/3}V^{1/2}/m_{\rm Pl}$. If so, then we would expect $\Delta T/T \sim 2\times 10^{-6}(E_{\rm infl}/10^{16}\,{\rm GeV})^2$, where $E_{\rm infl} = V^{1/4}$. Since gravitational waves contribute $\Delta T/T \lesssim 10^{-5}$, we infer $V^{1/4} \lesssim 2\times 10^{16}$ GeV. In fact, this is remarkably close to what a precise calculation gives.

Again, the gravitational-wave power spectrum is k independent only to the extent that H is constant during inflation. Most generally, ϕ rolls down the potential V, and so H decreases as inflation proceeds. There is thus a tensor spectral index,

$$n_t = \frac{d\ln P_{\rm gw}}{d\ln k} = -2\epsilon. \tag{39}$$

8 The Cosmic Microwave Background (CMB)

A proper treatment of the CMB requires relativistic cosmological perturbations. There are a number of ways to see this: (1) The redshift $z_{ls} \simeq 1100$ of the surface of last scatter implies Hubble velocities $v \simeq c$, while our Newtonian analysis assumed nonrelativistic velocities. (2) We saw earlier that a causally connected patch at the surface of last scatter subtends an angle of $\sim 1^\circ$. That means that different patches separated by more than 1° are superhorizon at the time of decoupling, again invalidating the Newtonian analysis. (3) The propagation of light through a perturbed Universe requires a general-relativistic analysis. Our discussion will therefore have to be heuristic, and we will have to do no more than summarize many results.

Temperature fluctuations are observed in the CMB at the level of $\Delta T/T \sim 10^{-5}$. These fluctuations are produced by primordial perturbations, manifest in several different ways. First is the *Sachs-Wolfe effect*. We have seen that inflation predicts primordial gravitational-potential fluctuations $\Phi(\vec{x}, t) \sim 10^{-5}$. According to general relativity, photons lose energy (get redshifted) as they climb out of gravitational-potential wells. Thus, the temperature of the CMB in a direction $\hat{\mathbf{n}}$ is $(\Delta T/T)(\hat{\mathbf{n}}) = \Phi(r_{ls}\mathbf{hatn})/3$, where r_{ls} is the distance to the surface of last scatter; that is, the temperature is determined by the depth of the primordial gravitational potential at the surface of last scatter. The Sachs-Wolfe effect is the dominant source of anisotropy on angular scales large compared with 1°. In a non-Einstein-de Sitter Universe, there can be a small amount of additional temperature fluctuations produced at the very largest angular scales through the *integrated Sachs-Wolfe (ISW) effect*, which arises due to the non-constancy of the gravitational-potential perturbations in a non-EdS Universe.

On smaller scales (when we are looking at regions of the sky that are in causal contact at the surface of last scatter), causal processes can occur. In particular, density perturbations can induce peculiar-velocity flows. Thus, on angular scales $\lesssim 1^\circ$, there will be additional temperature fluctuations produced by the movement of the gas at the surface of last scatter toward or away from us. Moreover, when CMB photons last scatter, the baryon-photon fluid still has considerable pressure, meaning that perturbations in the baryon-photon fluid oscillate (propagate as sounds waves) leading to the acoustic-peak structure in the power spectrum we will see below.

9 The CMB power spectrum

We recall that inflation makes statistical predictions about the distribution of matter in the Universe today, quantified by a matter power spectrum $P(k)$. Likewise, inflation makes statistical predictions about the primordial perturbations, and thus the temperature fluctuations, which are characterized by a CMB power spectrum C_ℓ, which is defined as follows. Suppose we measure the temperature fluctuation $(\Delta T/T)(\mathbf{hatn})$ as a function of position on the sky. Then