Very Brief Description of Finite-Temperature Field Theory

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These notes are intended to provide a schematic explanation for the description of a scalar field $\phi(t, \vec{x})$ in thermal equilibrium at temperature T in terms of a quantum field theory in imaginary time with periodic (in imaginary time) boundary conditions. You can find more details in textbooks on quantum field theory and/or statistical mechanics. The place I'm looking at right now is *Finite-Temperature Field Theory* by Joseph Kapusta (Cambridge University Press, Cambridge, 1989).

Suppose that at time $t = 0$ the quantum-mechanical states of the scalar field is $|\phi_a\rangle$. After a time t, it will evolve to $e^{-i\hat{H}t}|\phi_a\rangle$, where $\hat{H}$ is the Hamiltonian for the system. The probability to find the system in some state $|\phi_b\rangle$ after time t is thus $\langle\phi_b|e^{-i\hat{H}t}|\phi_a\rangle$. According to Feynman's path-integral approach to quantum mechanics, this amplitude can be written as

$$\langle\phi_b|e^{-i\hat{H}t}|\phi_a\rangle = \int_{\phi(t=0,\vec{x})=\phi_a(\vec{x})}^{\phi(t,\vec{x})=\phi_b(\vec{x})} [d\phi]\, e^{\frac{i}{\hbar}\int_0^t dt' \int d^3x \mathcal{L}}. \tag{1}$$

Here, $\mathcal{L}$ is the *classical* Lagrangian for the system, and the $[d\phi]$ denotes a path integral over all possible field configurations $\phi(t, \vec{x})$ that mediate between the initial (ϕ_a) and final (ϕ_b) configurations.

Let's now consider the system at finite temperature T. All of thermodynamic quantities in the system can be derived from the partition function,

$$Z = \mathrm{Tr} e^{-\hat{H}/kT} = \int d\phi_a \langle\phi_a|e^{-i\hat{H}t}|\phi_a\rangle, \tag{2}$$

where k is the Boltzmann constant. The trace of $e^{-\hat{H}/kT}$ is just a sum (the integral $\int d\phi_a$) over the states (i.e., $|\phi_a\rangle$) of the expectation value of $e^{-\hat{H}/kT}$. Comparing with Eq. (1) suggests that we can write the partition function as a path integral. To do so requires several steps: (1) We replace the time t by an imaginary time $\tau = it$. (2) We identify the temperature T with the (imaginary) time τ that has evolved through $T = \hbar/(k\tau)$. (3) We set the initial and final states that appear in Eq. (1) to be the same. (4) We carry out the integral over states ϕ_a in Eq. (2) by summing over all possible initial and final states (which are now identified) in Eq. (1). The result is

$$Z = \int_{\text{periodic}} [d\phi] e^{\int_0^{1/kT} d\tau \int d^3x \mathcal{L}}. \tag{3}$$

The "periodic" here means that the integration is over all field configurations in which $\phi(0, \vec{x}) = \phi(\tau = \hbar/kT, \vec{x})$; i.e., over all field configurations that return to their initial value after imaginary time $\tau = \hbar/kT$.

This is why a scalar field that lives in a spacetime that can be written in terms of an imaginary time with periodic boundary conditions is equivalent to a scalar field at finite temperature.