

General Relativity (Ph236a)

Problem Set 1

Due: October 3, 2006

Preview: Problem 1 is highly recommended. The first part should help you understand/review the idea of a metric in a familiar setting; the second part will make Christoffel symbols seem more digestible when we get to them in a few weeks. Problem 2 is something you should be able to do if you understand special relativity well. Problems 3 and 4 are important, as they will help you understand more complicated spacetime diagrams that we will get to later in the class. Problems 5 and 6 are cute exercises in special-relativistic kinematics, but should probably be given lower priority than the others if you have limited time.

1. **Euclidean space in cylindrical and spherical coordinates:** Consider ordinary three-dimensional Euclidean space. The simplest coordinates are Cartesian coordinates, x^i , for $i = 1, 2, 3$. However, one could also choose to use any other coordinate system, $y^i(x_j)$, for example, cylindrical or spherical coordinates. In Cartesian coordinates, the distance ds between x^i and $x^i + dx^i$ is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

- a. Show that in the new coordinate system, y^i , this is given by

$$ds^2 = g_{ij} dy^i dy^j,$$

and find g_{ij} . Do this only for spherical coordinates; if you're a glutton for punishment, work it out for cylindrical coordinates as well.

- b. Now consider force-free Newtonian motion in this space. You know that trajectories are straight lines. Show that in y^i coordinates, the Lagrangian is

$$L = \frac{1}{2} m g_{ij} \dot{y}^i \dot{y}^j,$$

where the dot denotes derivative with respect to time. Next, solve the Euler-Lagrange equations to show that the equations of motion are (i.e., $\ddot{x}^i = 0$)

$$\ddot{y}^i + \Gamma_{jk}^i \dot{y}^j \dot{y}^k = 0,$$

where

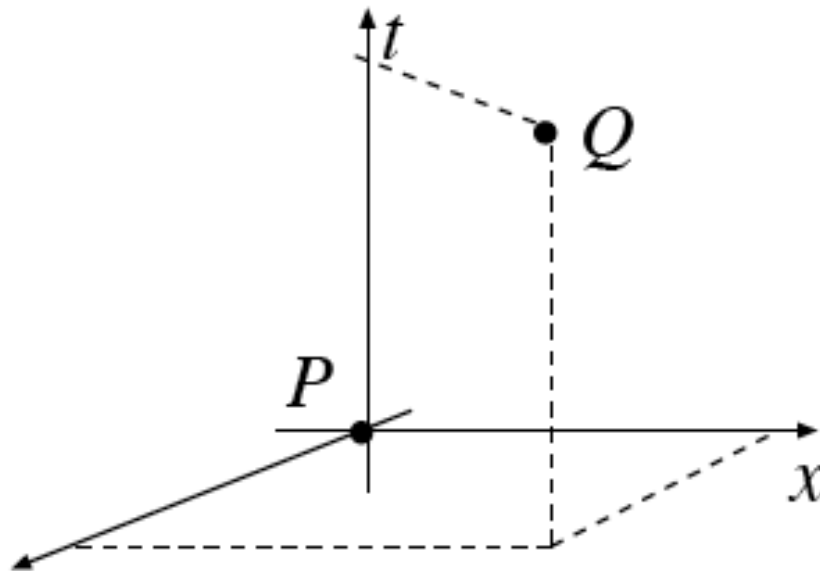
$$\Gamma_{jk}^i = \frac{1}{2} g^{il} (\partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{jk}),$$

and g^{ij} is the inverse metric, $g_{ij} g^{jk} = \delta_i^k$. Do *not* bother to calculate the components of Γ_{jk}^i .

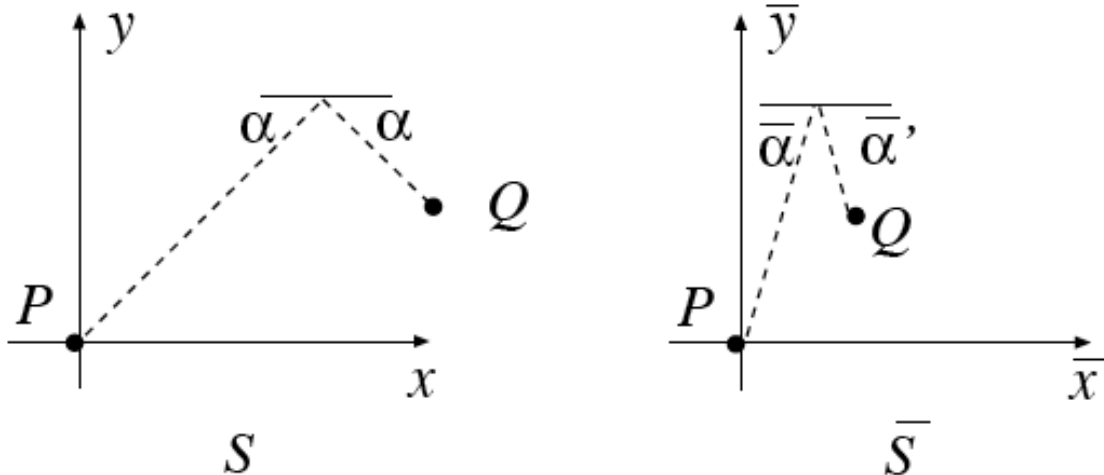
2. (Wald 1.1) **Car and Garage Paradox:** The lack of a notion of absolute simultaneity in special relativity leads to many supposed paradoxes. One of the most famous of these

involves a car and a garage of equal proper length. The driver speeds toward the garage and a doorman at the garage is instructed to slam the door shut as soon as the back end of the car enters the garage. According to the doorman, “the car Lorentz contracted and easily fitted into the garage when I slammed the door.” According to the driver, “the garage Lorentz contracted and was too small for the car when I entered the garage.” Draw a spacetime diagram showing the above events and explain what really happens. Is the doorman’s statement correct? Is the driver’s statement correct? For definiteness, assume that the car crashes through the back wall of the garage without stopping or slowing down.

3. **Proof of invariance of a timelike interval:** Consider two events \mathcal{P} and \mathcal{Q} in spacetime, with a timelike separation vector \vec{A} . Examine these events in two different reference frames \mathcal{S} and $\bar{\mathcal{S}}$ that move with speed v relative to each other. Choose the origins of the two frames’ spacetime coordinates to coincide and to be at the event \mathcal{P} and orient the spatial axes of the two frames so their relative motion is in the x direction and \mathcal{Q} lies in the x - y plane. The following diagram depicts this in a spacetime diagram drawn from the viewpoint of frame \mathcal{S} (note that the unlabeled axis is the y axis):



- a. Convince yourself that wherever may be the event \mathcal{P} and \mathcal{Q} , the origins and axes of the two frames can be adjusted as described above. The following experiment is a foundation for providing the invariance of the interval between \mathcal{P} and \mathcal{Q} . The experiment is sketched below in two purely spatial diagrams (time not shown) from the frames’ two different viewpoints:



A photon is emitted from \mathcal{P} and travels along a straight line in the x - y plane until it hits a mirror that reflects it; the photon then travels again in a straight line in the x - y plane, arriving at the event \mathcal{Q} . The position of the reflecting mirror is adjusted so the photon reaches the spatial location of \mathcal{Q} precisely at the time of \mathcal{Q} . The state of motion of the mirror is not important; the key thing is that, as seen in frame \mathcal{S} , the photon's direction of motion makes an angle α with the x axis that is the same before and after reflection (angle of incidence equals angle of reflection), as shown in the diagram. In the following, do *not* use the Lorentz-transformation equations. Assume that they have not yet been derived; we are working our way toward them, and our first step is to derive the invariance of the interval from the Principle of Relativity (i.e., there is no preferred inertial frame).

- b. Use the Principle of Relativity to show that the heights of the reflection point are the same in the two frames, $y_{\text{refl}} = \bar{y}_{\text{refl}}$, and that the angles of incidence and reflection are equal in the frame $\bar{\mathcal{S}}$, $\bar{\alpha} = \bar{\alpha}'$, just as they are equal in frame \mathcal{S} .
- c. Use the Principle of Relativity, the constancy of the speed of light, and simple geometric considerations to show that the interval between \mathcal{P} and \mathcal{Q} is the same as computed in the two reference frames:

$$-(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -(\Delta \bar{t})^2 + (\Delta \bar{x})^2 + (\Delta \bar{y})^2 + (\Delta \bar{z})^2.$$

4. For the two reference frames of the preceding problem, draw a spacetime diagram for the \bar{x} - \bar{t} plane from the viewpoint of frame $\bar{\mathcal{S}}$ (*not* \mathcal{S}). On this diagram, draw the x and t axes of frame \mathcal{S} . Use this diagram to prove the following results:
 - a. Events that are simultaneous as seen in frame \mathcal{S} are not simultaneous as seen in $\bar{\mathcal{S}}$. Which events occur first and which later?
 - b. An ideal clock at rest in frame \mathcal{S} appears, as seen from $\bar{\mathcal{S}}$, to tick abnormally slowly; and conversely, an ideal clock at rest in frame $\bar{\mathcal{S}}$ appears, as seen from \mathcal{S} to tick abnormally slowly. Use the geometry of the diagram to show that the slow-down factor is $\sqrt{1 - v^2}$.
 - c. An ideal rod at rest in frame \mathcal{S} appears, as seen from $\bar{\mathcal{S}}$, to be contracted by a factor $\sqrt{1 - v^2}$; and conversely, an ideal rod at rest in frame $\bar{\mathcal{S}}$ appears, as seen from \mathcal{S} , to be contracted by that same factor.

5. **Addition of velocities I:** If two frames move with 3-velocities \vec{v}_1 and \vec{v}_2 with respect to some inertial frame, show that their relative 3-velocity has the magnitude,

$$\vec{v} \cdot \vec{v} = \frac{|\vec{v}_1 - \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2}{|1 - \vec{v}_1 \cdot \vec{v}_2|^2}.$$

6. **Addition of velocities II:** A cart rolls on a long table with 3-speed v . A smaller cart rolls on the first in the same direction with 3-speed v relative to the second cart. A third cart rolls on the second cart in the same direction with 3-speed v relative to the second cart, and so on up to n carts. What is the 3-speed v_n of the n th cart in the rest frame of the table? Compute the limit of v_n as $n \rightarrow \infty$. (Hint: consider the “rapidity parameter” $\theta = \tanh^{-1} v$.)