General Relativity (Ph236b)
Problem Set 1
Due: 16 January 2007

Preview: Problem 1 asks you to integrate numerically the TOV equation of state for a semi-realistic neutron-star equation of state. Its a good problem, but it may also be time-consuming. You should try to at least understand the procedure, even if you don't get all the way through the numerics. Problem 2 asks for the relatively simple proof of Birkhoff's theorem. Problem 3 should be a good problem to insure that you really understand the derivation of the Tolman-Oppenheimer-Volkoff equation of state. Problem 4 is a straightforward exercise to show that the outer event horizon in a Kerr spacetime is in fact a horizon. Problem 5 considers the possibility of energy extraction from a charged black hole; there's good physics here, and this problem should receive high priority. Problem 6 considers some of the observable astrophysical implications of a rotating black hole and should be high priority if you're interested in astrophysical applications of GR.

1. (From Lee Lindblom) **Numerical models of neutron stars:** Neutron stars are configurations of cold matter at the endpoint of thermonuclear evolution having central densities roughly in the range $10^{13} \le \rho \le 10^{16}$ g/cm^3. In this density range (which is about the density of an atomic nucleus), the bulk of the material in the star can be roughly approximated as a degnerate Fermi gas of neutrons. The equation of state for such a gas takes the form,

$$p = \frac{3^{2/3}\pi^{4/3}}{5}\frac{\hbar^2}{m_n^{8/3}}\rho^{5/3} \simeq 5.38 \times 10^9 \frac{\text{dynes}}{\text{cm}^2}\left(\frac{\rho}{1\text{gm/cm}^3}\right)^{5/3},$$

where $\hbar$ is Planck's constant (divided by 2π) and m_n is the neutron mass.

   a. Re-express the equation of state in geometrical units, with p and ρ having dimensions cm^{-2}.

   b. Integrate the Oppenheimer-Volkoff equations numerically using Mathematica, Maple, or whatever else you prefer. Compute the total mass M and the total radius R for a sequence of central densities; e.g., $\rho_c = 10^{14}$, 3×10^{14}, 10^{15}, 3×10^{15}, and 10^{16} (and additional points in between, if you wish). Plot the resulting mass-radius relation (analogous to Fig. 6.1 of Wald) for stellar models based on this equation of state.

2. (Also from Lee) **Birkhoff's theorem:** In class we consider a spacetime that has spherical symmetry and argued that the metric for such a spacetime can be written in the form,

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

In class, we assumed further that the spacetime was static; i.e., that $\nu = \nu(r)$ and $\lambda = \lambda(r)$ were functions of r only (and not of time t). Now relax the staticity assumption: i.e., take $\nu = \nu(t, r)$ and $\lambda = \lambda(t, r)$ and then use the Einstein vacuum field equations to show that the only spherical symmetric vacuum spacetime is the Schwarzchild spacetime: i.e., that ν and λ are *in*dependent of t. This result, known as Birkhoff's theorem, implies that the gravitational field inside a nonrotating spherical shell is also of the Schwarchild form. It is sort of a general-relativistic generalization of Newton's theorem that spherical shells of matter give rise to no gravitational field inside them.

3. (Carroll, problem 5.2) **(2+1)-dimensional star:** Let's return to the static circularly-symmetric (2+1)-dimensional spacetime [or equivalently, a (3+1)-dimensional spacetime with cylindrical symmetry].
   a. Derive the Tolman-Oppenheimer-Volkoff equation for $(2+1)$ dimensions.
   b. Solve the (2+1)-dimensional TOV equation for a constant-density star. Find the pressure $p(r)$ and solve for the metric.
   c. Find the mass $M(r) = \int_0^{2\pi}\int_0^R \rho\, dr\, d\theta$ and the proper mass $\bar{M}(R) = \int_0^R \rho\sqrt{-g}\, dr\, d\theta$ for the solution in part (b).

4. (Carroll, problem 6.2) **Geodesics of Kerr spacetime:** Consider the orbits of massless particles, with affine parameter λ, in the equatorial plane of a Kerr black hole.
   a. Show that
   $$\left(\frac{dr}{d\lambda}\right)^2 = \frac{\Sigma^2}{\rho^4}[E - LW_+(r)][E - LW_-(r)],$$
   where $\Sigma^2 = (r^2+a^2)^2 - a^2\Delta(r)\sin^2\theta$, E and L are the conserved energy and angular momentum, and you have to find expressions for $W_\pm(r)$.
   b. Using this result, and assuming that $\Sigma^2 > 0$ everywhere, show that the orbit of a photon in the equatorial plane cannot have a turning point inside the outer event horizon r_+. This means that ingoing light rays cannot escape once they cross r_+, so it really is an event horizon.

5. (Carroll, problem 6.3) **Charged particles in a Reissner-Nordstrom spacetime:** In the presence of an electromagnetic field, a particle of charge e and mass m obeys
   $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{e}{m}F^\mu{}_\nu\frac{dx^\nu}{d\tau}.$$
   Imagine that such a particle is moving in the field of a Reissner-Nordstrom black hole with charge Q and mass M.
   a. Show that the energy
   $$E = m\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)\frac{dt}{d\tau} + \frac{eQ}{r},$$
   is conserved.
   b. Will a Penrose-type process work for a charged black hole? What is the maximum change δM in the black-hole mass?

6. (Carroll, problem 6.6) **Signals from astrophysical Kerr black holes:** Consider a Kerr black hole with an accretion disk, of negligible mass, in the equatorial plane. Assume

that particles in the disk follow geodesics (that is, ignore any pressure support). Now suppose that the disk contains some iron atoms (actually, in very highly-ionized states) that are being excited by some source of radiation. When the iron atoms de-excite they emit radiation with a known frequency ν_0 (which in practice is in the x-ray regime), as measured in their rest frame. Suppose we detect this radiation far from the black hole (and assume that we also lie in the equatorial plane). What is the observed frequency of photons emitted from the inner edge of the accretion disk, from both the approaching side and the receding side. Consider cases where the disk and the black hole are rotating in the same and in the opposite directions. Can we use these measurements to determine the mass and angular momentum of the black hole?