General Relativity (Ph236a)
Problem Set 2
Due: October 10, 2006

Preview: Problem 1 asks you to calculate components of the stress-energy tensor for a few configurations. Its a good one to do, as it is intended to put some meat on the concept of $T^{\mu\nu}$. Problem 2 is a cute exercise involving another special-relativistic "paradox." I'd rank it lower priority, but it is nice in that it is actually seen in a number of astrophysical settings. Problems 3 and 4 are cute exercises in special-relativistic dynamics of accelerated observers. The numerical answers are also kind of neat for science fiction fans, but they are not conceptually as important as the last two problems. Problem 5 is important, as it will be relevant for our discussion of spinning-frame experiments later in the year. Problem 6 is also important for the same reason; it deals with the relativistic motion of a gyroscope.

1. **Components of Stress-Energy Tensor:** Calculate the components of the stress-energy tensors (in an inertial frame $\mathcal{O}$) in the following systems: (a) A group of particles all moving with the same velocity $\mathbf{v} = \beta \mathbf{e}_x$, as seen in $\mathcal{O}$. Let the rest-mass density of these particles be ρ_0, as measured in their comoving frame. Assume a sufficiently high density of particles to enable treating them as a continuum. (b) A ring of N similar particles of mass m rotating counter-clockwise in the $x-y$ plane about the origin of $\mathcal{O}$, at a radius a from this point, with an angular velocity ω. The ring is a torus of circular cross-section of radius $\delta a \ll a$, within which the particles are uniformly distributed with a high-enough density for the continuum approximation to apply. Do not include the stress-energy of whatever forces keep them in orbit. (Part of the calculation will relate ρ_o of part (a) to N, a, ω, and δa.) (c) Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius a. The particles do not collide or interact in any way.

2. (Carroll, problem 1.4) **"Superluminal" motion:** Projection effects can trick you into thinking that an astrophysical object is moving "superluminally." Consider a quasar that ejects gas with speed v at an angle θ with respect to the line of sight of the observer. Projected onto the sky, the gas appears to travel perpendicular to the line of sight with angular speed $v_{\rm app}/D$, where D is the distance to the quasar and $v_{\rm app}$ is the apparent speed. Derive an expression for $v_{\rm app}$ in terms of v and θ. Show that, for appropriate values of v and θ, $v_{\rm app}$ can be greater than 1.

3. (MTW, 6.1-2) **Interstellar travel:**
   a. Compute the time required to travel in rocket ship to the Galactic center assuming a uniform acceleration of 1000 $\mathrm{cm}^2/\mathrm{sec}$, the acceleration of gravity at the Earth's surface.
   b. What fraction of the initial mass of the rocket can be payload for the journey? Assume a futuristic rocket that converts rest mass into radiation and ejects it with 100% efficiency opposite to the direction of motion.

4. (MTW 6.3) **Twin paradox:**
   a. Show that the timelike world line connecting two timelike-separated events $\mathcal{A}$ and $\mathcal{B}$ with the *longest* proper time is the unaccelerated one.
   b. Suppose that two twins travel from $\mathcal{A}$ to $\mathcal{B}$. The first twin, Jane, likes to take it easy; she prefers to travel along an intertial trajectory between $\mathcal{A}$ and $\mathcal{B}$. She also lives a good long life; it takes her 100 years to do so. The second twin, Alice, likes to get out and see things, but she can't tolerate an acceleration more than one Earth gravity, 1000 cm^2/sec. Assuming she also starts at $\mathcal{A}$, what is the shortest proper time that can elapse in her frame if she returns to re-join Jane at $\mathcal{B}$?

5. (MTW 6.8) **Observer with rotating tetrad:** An observer moving alolng an arbitrarily accelerated world line chooses *not* to Fermi-Walker his orthonormal tetrad. Instead, he allows it to rotate. The antisymmetric rotation tensor $\Omega^{\mu\nu}$ that enters the transport law,

$$\frac{de^{\mu}_{\alpha'}}{d\tau} = -\Omega^{\mu\nu} e_{\alpha'\;\mu},$$

splits into a Fermi-Walker part plus a spatial rotation, $\Omega^{\mu\nu} = \Omega^{\mu\nu}_{(FW)} + \Omega^{\mu\nu}_{(SR)}$, where $\Omega^{\mu\nu}_{(FW)} = a^{\mu}u^{\nu} - a^{\nu}u^{\mu}$ and $\Omega^{\mu\nu}_{(SR)} = u_{\alpha}\omega_{\beta}\epsilon^{\alpha\beta\mu\nu}$, and ω^{μ} is a four-vector orthongal to the four-velocity: $\omega^{\mu}u_{\mu} = 0$.
   a. The observer chooses his time basis vector to be $e^{\mu}_{0'} = u^{\mu}$. Show that this choice is permitted by the transport law implied by the above choice for $\Omega^{\mu\nu}$.
   b. Show that $\Omega^{\mu\nu}_{(SR)}$ produces a rotation in the plane perpendicular to u^{μ} and ω^{μ}—i.e., that $\Omega^{\mu\nu}_{(SR)}u_{\nu} = 0$ and $\Omega^{\mu\nu}_{(SR)}\omega_{\nu} = 0$.
   c. Suppose that the accelerated observer Fermi-Walker transports a second orthonormal tetrad $e^{\mu}_{\alpha''}$. Show that the space vectors of his first tetrad rotate relative to those of his second tetrad with angular velocity equal to ω. [Hint: At a moment when the tetrads coincide, show that (in three-dimensional notation, referring to the 3-space orthogonal to the observer's world line):

$$\frac{d(\vec{e}_{j'} - \vec{e}_{j''})}{d\tau} = \vec{\omega} \times \vec{e}_{j'}.$$

   d. The observer uses the prescription discussed in class [this is MTW's Eq. (6.16)] to set up local coordinates based on his rotating tetrad as for his Fermi-Walker tetrad. Pick an event $\mathcal{Q}$ on the observer's world line, set $\tau = 0$ there, and choose the original inertial frame of prescription (6.16) so (1) it comoves with the accelerated observer at $\mathcal{Q}$, (2) its origin is at $\mathcal{Q}$, and (3) its axes coincide with the accelerated axes at $\mathcal{Q}$. Show that these conditions translate into $z^{\mu} = 0$ and $e^{\mu}_{\alpha'}(0) = e_{\alpha}$.
   e. Show that near $\mathcal{Q}$, the prescription [MTW's Eq. (6.16)] for setting up rotating accelerated coordinates reduces to:

$$x^0 = \xi^{0'} + a_k\xi^{k'}\xi^{0'} + O([\xi^{\alpha'}]^3);$$

$$x^j = \xi^{j'} + \frac{1}{2}a^j(\xi^{0'})^2 + \epsilon^{jkl}\omega^k\xi^{l'}\xi^{0'} + O([\xi^{\alpha'}]^3).$$

f. A freely moving particle passes through the event $\mathcal{Q}$ with ordinary 3-velocity $\vec{v}$ measured in the inertial frame. By transforming its straight world line $x^j = v^j x^0$ to the accelerated, rotating coordinates, show that its coordinate velocity and acceleration there are:

$$(d\xi^{j'}/d\xi^{0'})_{\text{at } \mathcal{Q}} = v^j;$$

$$(d^2\xi^{j'}/d(\xi^{0'})^2)_{\text{at } \mathcal{Q}} = -a^j - 2\epsilon^{jkl}\omega^k v^l + 2v^j a^k v^k.$$

Here the first term is the inertial acceleration (the acceleration of the center of mass), the second term is the Coriolis acceleration, and the third term is a relativistic correction to inertial acceleration.

6. (MTW 6.9) **Thomas precession:** Consider a spinning body (gyroscope, electron,...) that accelerates because forces act at its center of mass. Such forces produce no torques; so they leave the body's intrinsic angular-momentum vector S^μ unchanged, except for the unique rotation in the velocity-acceleration plane required to keep S^μ orthogonal to the four-velocity u^μ (note that the spin four-vector S^μ is required to have $S^\mu u_\mu = 0$ so that its components in the frame defined by $u^\mu = (1, 0, 0, 0)$ are the components of the usual spin three-vector $\vec{S}$). The body Fermi-Walker transports its angular momentum (no rotation in planes other than those defined by u^μ and the acceleration a^μ):

$$dS^\mu/d\tau = (u^\mu a^\nu - u^\nu a^\nu)S_\nu.$$

This transport law applies to a spinning electron that moves in a circular orbit of radius r around an atomic nucleus. As seen in the laboratory frame, the electron moves in the x-y plane with constant angular velocity ω. At time $t = 0$, the electron is at $x = r$, $y = 0$, and its spin (as treated classically) has components,

$$S^0 = 0, \qquad S^x = \frac{\hbar}{\sqrt{2}}, \qquad S^y = 0, \qquad S^z = \hbar.$$

a. Calculate the subsequent behavior of the spin $S^\mu(t)$ as a function of laboratory time t.

b. Write an expression for $S^x + iS^y$ as a function of time t. Show that it describes precession in a retrograde direction, and determine the angular velocity ω_{Thomas} of this precession.