General Relativity Ph236b
Problem Set 2
Due: In class, January 23, 2007

Preview: Problem 1 is a simple exercise to show that the two forms of the Friedmann equation lead to identical dynamics, and Problem 2 gives a Newtonian derivation of these equations. Problem 3 investigates the equivalence between an empty (i.e., $\rho = 0$) FRW Universe and Minkowski spacetime. Problem 4 derives the cosmological redshift and peculiar-velocity decay. Problem 5 has you calculate the age of the Universe. Problem 6 deals with the representation of de Sitter space as an FRW Universe. Problems 4, 5, and 6 should probably be given highest priority, although these should all be relatively simple problems.

1. **Two forms of the Friedmann equations:** Show that the two forms of the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2},$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3}(\rho + 3p),$$

lead to identical dynamics. Do *not* assume that the matter is only pressureless.

2. **Newtonian derivation of Friedmann equations:** The point of this problem is to show that the Friedmann equations for a Universe consisting of nonrelativistic matter can be derived with a Newtonian analysis. To do so, you consider an infinite homogeneous universe and then draw a sphere of radius a around an arbitrary point. You then argue that the universe outside the sphere can be considered as a sequence of concentric spherical shells and therefore do not impact the dynamics of the sphere. The dynamics of this universe then reduce to the dynamics of an expanding homogeneous sphere. Show that the equation of motion for the radius a of a homogeneous sphere are the Friedmann equations (both forms). Once you've done that, consider whether the same analysis can be used for relativistic matter by replacing the gravitating mass density ρ by $\rho + 3p$, where p is the pressure.

3. **Milne spacetime:** Find the coordinate transformation that shows that the Milne spacetime (i.e., the FRW spacetime with Friedmann equation $H^2 \propto a^{-2}$) is equivalent to a Minkowski spacetime. Explain what is going on.

4. **Cosmological redshift and peculiar-velocity decay:** Derive the photon redshift and the peculiar-velocity decay for a massive particle in a FRW spacetime from the geodesic equation or by finding a conserved quantity associated with a Killing vector.

5. **Age of the Universe:** The age t_0 of the Universe in the standard cosmological model depends on the current value of the Hubble parameter, $H_0 = 100\,h$ km/sec/Mpc, as well as on Ω_m, the current nonrelativistic-matter density in units of the critical density. In class, we showed that if $\Omega_m = 1$ and $\Omega_\Lambda = 0$, then $t_0(h) = 6.7\,h^{-1}$ Gyr.
   a. Generalize this result and derive an expression for the age of the Universe for $\Omega_m > 1$ and for $\Omega_m < 1$, both for $\Omega_\Lambda = 0$. (This shouldn't be too tricky—the answers are in the books. But still, you should derive the equations yourself.) Then, plot contours for t_0 =10 Gyr, 13 Gyr, and 17 Gyr on the $\Omega_m - h$ plane. You may do this either by sketching the contours by hand, or you may generate such a plot with Mathematica, C, Fortran, or anything else, if you're so inclined.
   b. Then, make the analogous plots, but for $\Omega_m + \Omega_\Lambda = 1$ (and restricting to $\Omega_m < 1$).
   c. Current measurements indicate $\Omega_m h^2 = 0.127 \pm 0.10$ and $h = 0.73 \pm 0.03$. What range of ages is consistent with these data (assuming $\Omega_m + \Omega_\Lambda = 1$). Suppose the matter density is in the range $0.2 < \Omega_m < 0.4$ and the Hubble parameter is in the range $0.55 < h < 0.85$ (the current limits are Stellar astrophysicists believe that the oldest stars are around 10–20 Gyr. Show on your plots correct value for the age of the Universe is somewhere around 14 Gyr, your plots should show you for which values of Ω_m and h there might be consistency.

6. **de Sitter spacetime as an FRW Universe:** In problem 6 of Problem Set 8 from the first quarter, you showed that the spacetime for a constant-curvature spacetime with nonzero cosmological constant Λ is the de Sitter spacetime,

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right) dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\Omega^2.$$

   a. Find the coordinate transformation that allows us to write this spacetime as a FRW spacetime, of the form,

$$ds^2 = -d\tau^2 + a^2(\tau)(dx^2 + dy^2 + dz^2),$$

   and find the function $a(\tau)$.
   b. Solve the geodesic equation for comoving observers (x^i =constant) to find the affine parameter as a function of t. Show that geodesics reach $t = -\infty$ in finite affine parameter, demonstrating that these coordinates fail to cover the entire manifold.