General Relativity (Ph236a)
Problem Set 3
Due: October 17, 2006

Preview: Problem 1 is essential; it defines the Riemann normal coordinates, or the coordinates for a locally inertial frame. Problems 2–5 are exercises intended to develop a facility with vector fields and their dual one-form fields. Problem 6 is a straightforward exercise to develop some facility with metrics; it also illustrates some features of spinning spacetimes that will show up later when we deal with gravitomagnetism and with spinning black holes.

1. **Riemann normal coordinates, or a locally inertial frame:** Show that a coordinate system can always be chosen so that at a given point p, the metric $g_{\mu\nu}$ is canonical and $\partial_\rho g_{\mu\nu} = 0$.

2. (Wald 3.3) **Commutator of vector fields:**
   a. Verify that the commutator,
   $$[v, w](f) = v[w(f)] - w[v(f)],$$
   where v and w are smooth vector fields, and f is a function, satisfies the linearity and Leibnitz properties, and hence defines a vector field.
   b. Let X, Y, and Z be vector fields on a manifold. Verify that their commutator satisfies the Jacobi identity,
   $$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$
   c. Let $Y_1, ..., Y_n$ be vector fields on an n-dimensional manifold such that at each point p on the manifold, they form a basis of the tangent space V_p. Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining the functions $C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha}$ by
   $$[Y_\alpha, Y_\beta] = \sum_\gamma C^\gamma{}_{\alpha\beta} Y_\gamma.$$
   Use the Jacobi identity to derive an equation satisfied by $C^\gamma{}_{\alpha\beta}$.

3. (Wald 3.4) **More on commutators:**
   a. Show that in any coordinate basis, the components of the commutator of two vector fields v and w are given by
   $$[v, w]^\mu = \sum_\nu \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right).$$
   b. Let $Y_1, ..., Y_n$ be as in problem 2(c). Let $Y^{1*}, ..., Y^{n*}$ be the dual basis. Show that the components $(Y^{\gamma *})_\mu$ of $Y^{\gamma *}$ in any coordinate basis satisfy
   $$\frac{\partial (Y^{\gamma *})_\mu}{\partial x^\nu} - \frac{\partial (Y^{\gamma *})_\nu}{\partial x^\mu} = \sum_{\alpha,\beta} C^\gamma{}_{\alpha\beta} (Y^{\alpha *})_\mu (Y^{\beta *})_\nu.$$

[Hint: Contract both sides with $(Y_\sigma)^\mu (Y_\rho)^\nu$. Also, be careful to distinguish between indices that label the vector fields from indices that label their components.]

4. (MTW, Problem 9.6) **Practice with dual bases:** In a three-dimensional space with spherical coordinates r, θ, and ϕ, it is often useful to use, instead of the coordinate basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\phi$, the orthonormal basis,

$$e_{\hat{r}} = \frac{\partial}{\partial r}, \qquad e_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial\theta}, \qquad e_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

   a. What is the one-form basis $\{\omega^{\hat{r}}, \omega^{\hat{\theta}}, \omega^{\hat{\phi}}\}$ dual to this tangent-vector basis?
   b. On the sphere of unit radius, draw pictures of the bases $\{\partial/\partial r, \partial/\partial\theta, \partial/\partial\phi\}$, $\{e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}\}$, $\{dr, d\theta, d\phi\}$ (the one-form basis dual to the coordinate tangent-vector basis), and $\{\omega^{\hat{r}}, \omega^{\hat{\theta}}, \omega^{\hat{\phi}}\}$.

5. (MTW, Problems 9.7-8) **Commutators for Euclidean space in spherical coordinates:**
   a. Evaluate the commutator of the two vector fields $e_{\hat{\theta}} = (1/r)\partial/\partial\theta$ and $e_{\hat{\phi}} = (1/r\sin\theta)\partial/\partial\phi$. Use this to evaluate the components of the tensor $C^i{}_{jk}$ (of problems 2 and 3) for this two-dimensional vector space.
   b. In Cartesian coordinates of a 3-dimensional Euclidean space, define three "angular-momentum operators" $L_i = \epsilon_{ijk} x^j \partial_k$. Evaluate the components of the tensor $C^i{}_{jk}$ for these vector fields.

6. (Wald 2.8b+) **Rotating coordinates:** The line element of special relativity is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

   a. Find the line element in rotating coordinates, defined by

$$\begin{aligned} t' &= t, \\ x' &= (x^2 + y^2)^{1/2}\cos(\phi - \omega t), \\ y' &= (x^2 + y^2)^{1/2}\sin(\phi - \omega t), \\ z' &= z, \end{aligned}$$

   where $\tan\phi = y/x$.
   b. Re-write the line element in terms of cylindrical coordinates r' and ϕ' defined by $x' = r'\cos\phi'$ and $y' = r'\sin\phi'$.