General Relativity Ph236b

Problem Set 3

Due: In class, January 30, 2007

Preview: These should all be pretty straightforward problems. I would start with Problems 2, 4, and 6 (which deal with the most common classical cosmological tests) if you have limited time.

1. **FRW spacetimes:** Prove that the only three homogeneous and isotropic spacetimes are the three FRW spacetimes.

2. **Luminosity distances:** Astronomers measure the brightness of sources on a logarithmic *apparent magnitude* scale, where the apparent magnitude is defined to be $m = -2.5 \log f + \text{constant}$, and f is the flux, and the logarithm is base 10. The luminosity of the source is similarly measured on a logarithmic *absolute luminosity* scale, where the absolute luminosity is $M = -2.5 \log L + \text{constant}$. The constants are chosen so that the *distance modulus* is $m - M = 5 \log(r/10\,\text{pc})$. If there is a *standard candle*, an object of known luminosity L (and thus known M, then measurement of its apparent magnitude m determines the luminosity distance. Suppose now that observers measure the distance modulus of supernovae (assumed to be standard candles) at redshifts $z = 0.5$ and $z = 1$. Calculate the luminosity distances at these two redshifts for (i) an Einstein-de Sitter Universe ($\Omega_m = 1$, $\Omega_\Lambda = 0$), (ii) a flat Universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$, and (iii) an open Universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0$. Determine the differences in the distance moduli between these three cosmologies, both at redshifts $z = 0.5$ and $z = 1$.

3. **Thermal relations in expanding Universe:** Consider a gas of massless particles in thermal equilibrium, and with no particle-antiparticle asymmetry ($\mu = 0$).
   a. Show that for adiabatic changes, $dp/dT = (\rho + p)/T$.
   b. How is this equation changed if $\mu \neq 0$?
   c. Does this relation hold if there are massive particles?

4. **Mass, sizes, and age of the Universe:** Determine (a) the physical size of the currently observable Universe, (b) the mass enclosed within the horizon, and (c) the age of the Universe, when the temperature of the Universe was (i) 10^{16} GeV (a GUT phase transition), (ii) 100 GeV (the electroweak phase transition), (iii) 100 MeV (QCD phase transition), (iv) MeV (big-bang nucleosynthesis), and (v) an eV (decoupling).

5. **Conformal time:** Consider the conformal time η defined by $d\eta = dt/a(t)$. (a) Show that $\eta \propto a^{1/2}$ in a matter-dominated Universe and $\eta \propto a$ in a radiation-dominated Universe. (b) Consider a Universe with only matter and radiation, with matter-radiation equality at a_{eq}. Show that

$$\eta = \frac{2}{\sqrt{\Omega_m H_0^2}} \left[\sqrt{a + a_{\text{eq}}} - \sqrt{a_{\text{eq}}}\right] .$$

(c) Consider a de-Sitter Universe, with $a(t) \propto e^{Ht}$. What is the conformal time, as a function of scale factor a in this Universe?

6. (Carroll, problem 8.5) **Angular-diameter distances:** In a flat spacetime, objects of fixed physical size subtend smaller and smaller angles as they are further and further away; in an expanding universe, this is not necessarily the case. Consider the angular size $\theta(z)$ of an object of physical size L at redshift z. Find the redshift at which $\theta(z)/L$ is minimized for (a) a de-Sitter Universe ($\Omega_m = 1$, $\Omega_\Lambda = 0$) and for a Universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. If all galaxies are at least 10 kpc across (and always have been), what is the minimum angular size of a galaxy in these two universes? Express your result both in terms of H_0, and then plug in $H_0 = 70$ km/sec/Mpc.