General Relativity Ph236c
Problem Set 3
Due: In class, April 24, 2007

Suggested Reading: Carroll, 8.8; Kolb and Turner, Ch. 8; Dodelson, Ch. 6; Liddle and Lyth, *Cosmological Inflation and Large-Scale Structure*, Ch. 3; Peebles, Ch. 17.

Preview: This should be an interesting problem set; in it, you will work out several possibilities, many relevant to current research, for inflation, dark energy, and dark matter. The problems themselves are technically pretty simple, but together they cover quite a bit of territory. Problem 1 is a straightforward exercise in which you will show that a rolling scalar field (i.e., a scalar field in a kinetic-energy–dominated phase) acts like matter with pressure $p = \rho$. Problem 2 is a pretty involved problem in which you will work out pretty fully the phenomenological consequences of a particular model for inflation. Problem 3 works through a particularly intriguing quintessence model (i.e., a scalar-field model for negative-pressure dark energy in the Universe today) in which the dark-energy density tracks that of the dominant component (e.g., radiation or matter) of the cosmological energy density. Problem 4 is an order-of-magnitude calculation that shows that magnetic monopoles produced at a GUT phase transition should overwhelm the density of the Universe today (if there were no inflation); this reproduces a calculation that John Preskill was the first to do around 1980. Problem 5 shows that oscillations in an anharmonic scalar-field potential can give rise to exotic equations of state. Problem 6 has you calculate the flux of gamma rays produced by annihilation of WIMPs in the Galactic halo.

1. **A $w = 1$ equation of state from a rolling scalar field.** Consider a massless scalar field; i.e., a scalar field $\phi(\vec{x}, t)$ whose potential-energy density is $V(\phi) = 0$. Now suppose that this scalar field is initially rolling, so $\dot{\phi} \neq 0$, and that the kinetic-energy density associated with this rolling dominates the energy density of the Universe. Show from the stress-energy tensor $p = \rho$ for this type of matter. Show that this implies that $\rho \propto a^{-6}$, where a is the scale factor, in two ways: (1) by recalling how the energy density of matter with an equation of state $p = w\rho$ scales with a; and (2) by solving the equation of motion for ϕ in an expanding Universe. (This should be a very simple problem.)

2. (From LL 3.7) **Phenomenology of $\lambda\phi^4$ inflation.** Consider $V(\phi) = \lambda\phi^4$, where λ is the self-coupling. Assume that the field rolls toward $\phi = 0$ from the positive side. Calculate the value of ϕ where each of the slow-roll conditions (i.e., $\epsilon \ll 1$ and $\eta \ll 1$) first break down. Do they break down at the same place? Assuming that inflation ends when $\epsilon = 1$, calculate the number of e-foldings of inflation that occur for an initial value ϕ_i. Demonstrate that the slow-roll solutions with $\phi = \phi_i$ and $a = a_i$ at $t = t_i$ are

$$\phi = \phi_i \left[-\sqrt{\frac{32\lambda M_{\rm Pl}^2}{6}}(t - t_i) \right],$$

$$a = a_i \exp\left(\frac{\phi_i^2}{8M_{\rm Pl}^2}\left\{1-\exp\left[-\sqrt{\frac{64\lambda M_{\rm Pl}^2}{3}}(t-t_i)\right]\right\}\right).$$

Use the solution for ϕ to calculate the time that inflation ends. Demonstrate that the number of e-foldings calculated using the solution for a is the same as that which you calculated above. Expand the solution for a at small $t - t_i$ to demonstrate that the inflation is approximately exponential in the initial stage. Calculate the time constant κ [from $a \sim \exp(\kappa t)$] and demonstrate that it equals the (slow-roll) Hubble parameter during inflation.

3. **Tracker field.** Consider a scalar field that rolls down a potential-energy density $V(\phi) = V_0 e^{-\phi/\phi_0}$. Now suppose that the energy density of the Universe is dominated by ordinary non-relativistic matter (so $a \propto t^{2/3}$), and that the energy density of the rolling scalar field is negligible compared with the non-relativistic matter. Show that there is a solution to the scalar-field equation of motion such that the energy density $\rho_\phi = (1/2)\dot{\phi}^2 + V(\phi)$ of the scalar field scales as $\rho_\phi \propto a^{-3}$, the same as the ordinary matter. Does the same thing happen if the energy density of the Universe is dominated by relativistic matter? This is the basis for the "tracker-field" solutions that have been discussed in the literature recently.

4. **The monopole problem.** Calculate the relic density of magnetic monopoles, assuming that there is one GUT-mass ($\sim 10^{15}$ GeV) monopole produced per Hubble volume at the GUT phase transition ($T \sim 10^{15}$ GeV). You should get an unreasonably large number. There is a bound $\Omega_{\rm monopole} \lesssim 10^{-6}$ (the Parker bound) to the relic density of magnetic monopoles in the Universe today. Calculate the number of e-folds of inflation after the GUT transition required to solve the monopole problem.

5. **Anharmonic scalar-field oscillations.** In class we argued that if we have a real scalar field ϕ with a quadratic potential $V(\phi) = (1/2)m^2H^2$, and if $m \gtrsim H$ (implying that the oscillation frequency is large than the expansion rate), then coherent oscillations of the scalar field imply that the pressure $p = 0$ when averaged over an oscillation cycle and thus that the energy density $\rho \propto a^{-3}$. Now consider oscillations in a potential $V(\phi) = c|\phi|^n$, where c is a constant. Show that coherent oscillations in such a potential give rise to an energy density that decays as $\rho \propto a^{-\alpha}$, and determine α. Of course, you should recover $\alpha = 3$ for $n = 2$. What value of n is required to produce $\alpha = 4$ (i.e., radiation)? Can you think of a physical argument that justifies your result? Likewise, is there a value of n that produces $\alpha = 0$? Can you explain this result in physical terms?

6. **WIMPs in the Galactic halo.** This problem has nothing to do with inflation, but goes back to weakly-interacting massive particle (WIMP) dark matter. Suppose the Galactic halo has a density distribution,

$$\rho(r) = \rho_0 \frac{r_0^2 + a^2}{r^2 + a^2},$$

where $\rho_0 = \rho(r_0)$ is the local halo density, $r_0 \simeq 8.5$ kpc is our distance from the Galactic center, and $a \simeq 4$ kpc is the core radius. Suppose also that the halo is made out of

WIMPs χ with masses m_χ =10s–1000s GeV which have a cross section for annihilation to two photons $\sigma_{\chi\chi\to\gamma\gamma}$. (a) If ψ is the angle some given line of sight makes with the Galactic center (i.e., $\psi = 0$ is toward the Galactic center and $\psi = \pi$ is away from the Galactic center), calculate the flux of gamma rays observed along the line of sight ψ. (b) What is the energy of these photons. (c) The WIMP χ may annihilate to photons via a diagram such as that below, where the particles in the loops are quarks q and squarks $\tilde{q}$, and the squark mass is comparable to the WIMP mass. Give an order-of-magnitude estimate of this annihilation cross section and estimate the flux of photons from WIMP annihilation.