General Relativity (Ph236a)
Problem Set 4
Due: October 24, 2006

Preview: Problem 1 is intended to help you understand more deeply the derivation and meaning of the covariant derivative and the connection that we use in general relativity, by considering an alternative that we do not use. Problem 2 is a pretty straightforward exercise that illustrates what an affine parameter is. Problem 3 is a very straightforward exercise to familiarize you with the Christoffel symbols and derivative operators in a simple setting. If tedious algebra is not your thing, you can take this opportunity to familiarize yourself with algebraic-manipulation software for GR. Problem 4 is an important and probably enjoyable problem, as it actually deals (already!) with real physics. Problem 5 is intended to give you some practice with metrics and curvature and to preview the metric we use in cosmology. Problem 6 deals with the Rindler spacetime, which we discussed in class.

1. (Wald 3.1) **Connection with non-zero torsion:** When we introduced the covariant derivative ∇_μ in class (and in the books), we chose it so that it was torsion free. This condition, as well as a few others, gave rise to the Christoffel symbols that are used to relate the partial derivative ∂_μ to the covariant derivative. In this problem, you will consider a derivative operator that arises when the torsion-free condition is dropped.
   a. Show that the exists a tensor $T^\rho_{\mu\nu}$ (the *torsion tensor*) such that for all functions f, we have $\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = -T^\rho_{\mu\nu} \nabla_\rho f$.
   b. Show that for any smooth vector fields X^μ and Y^μ, we have

   $$T^\rho_{\mu\nu} X^\mu Y^\nu = X^\mu \nabla_\mu Y^\rho - Y^\mu \nabla_\mu X^\rho - [X, Y]^\rho.$$

   c. Given a metric $g_{\mu\nu}$, show that there exists a unique derivative operator ∇_μ with torsion $T^\rho_{\mu\nu}$ such that $\nabla_\rho g_{\mu\nu} = 0$. Write the connection for this derivative operator in terms of the metric, its ordinary (i.e., ∂_μ) derivatives, and the torsion $T^\rho_{\mu\nu}$.

2. (Wald 3.5) **Non-affine parameter for geodesic equation:**
   a. Show that any curve that satisifies

   $$\frac{d^2 x^\mu}{d\alpha^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\alpha} \frac{dx^\sigma}{d\alpha} = f(\alpha) \frac{dx^\mu}{d\alpha},$$

   can be re-parameterized by another parameter $\lambda(\alpha)$ so that the usual geodesic equation,

   $$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0,$$

   is satisfied. A parameterization λ for the curve that allows the geodesic equation to be written in this form is known as an *affine parameter*.
   b. Let λ be an affine parameter of a geodesic. Show that all other affine parameters of the geodesic take the form $a\lambda + b$, where a and b are constants.

3. **Christoffel symbols for Euclidean space in spherical coordinates:** The metric for three-dimensional Euclidean space in spherical coordinates is $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$.
   a. Calculate the Christoffel components Γ^i_{jk} in this coordinate system. You can do this in one of two ways: (i) By brute force with pencil and paper, or (ii) with one of the algebraic-manipulation software for general relativity (see the class home page).
   b. Find for these coordinates (i) the components of $\nabla_i f$, where f is a function; (ii) the divergence $\nabla_i v^i$ of a vector field; (iii) the Laplacian $\nabla^i \nabla_i$; and (iv) the volume element dV.

4. (Carroll, problem 3.6+) **Terrestrial clocks:** Later on this quarter, we will see that a good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

   where $\Phi = GM/r$ is the Newtonian gravitational potential Here, G is Newton's constant and M is the mass of the earth.
   a. Show that for phenomena at or near the surface of the Earth, Φ is small. What this means is that the departure of the spacetime metric from the Minkowski spacetime are small. The exact spacetime differs from this one by Φ.
   b. Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?
   c. Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?
   d. How much proper time elapses while the satellite at radius R_1 (skimming along the surface of the Earth, neglecting air resistance) completes in one orbit? Work only to first order in Φ. Plug in the actual numbers for the radius and mass of the Earth to get an answer *in seconds*. How does this number compare to the proper time elapsed on the clock stationary on the surface? Are differences of this magnitude measurable with modern (atomic) clocks?

5. **Curvature for (2+1)-d Friedmann-Robertson-Walker metric:** As we will see in the winter quarter, the spacetime metric for an expanding three-dimensional (2 space and 1 time) Universe can be written,

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right], \\ &= g_{\mu\nu}dx^\mu dx^\nu, \\ &= -dt^2 + a^2(t)h_{ij}x^i x^j, \end{aligned}$$

   where k is a parameter that takes on the values 0 or ± 1.
   a. Calculate the *spatial*, the Ricci scalar (scalar curvature) for the spatial part h_{ij} of the metric for the three values of k.

b. Calculate the *spacetime curvature*, the Ricci scalar (scalar curvature) for the full metric $g_{\mu\nu}$.

6. **The Rindler spacetime:** Consider the two-dimensional spacetime metric $ds^2 = -x^2\,dt^2 + dx^2$, with coordinate ranges $-\infty < t < \infty$ and $0 < x < \infty$. It looks like something special happens at $x = 0$ in this spacetime, as the inverse metric is singular as $x = 0$. You will see in the following that this is just a coordinate (rather than physical) singularity of the spacetime.
   a. Find the null geodesics [curves $x(t)$ with $g_{\mu\nu}(dx^\mu/\tau)(dx^\nu/d\tau) = 0$] for this spacetime.
   b. Calculate the Riemann tensor for this spacetime.
   c. The results of part (b) should lead you to suspect that this spacetime is the same as Minkowski spacetime. Find the coordinate transformation that demonstrates this.
   d. Find a coordinate transformation that puts the metric in the form $ds^2 = e^{v-u}\,du\,dv$, and show via explicit calculation that the curvature of this metric agrees with those for parts (c) and (d).