General Relativity Ph236b

Problem Set 4

Due: In class, February 6, 2007

Preview: This should be a fun problem set. Problem 1 calculates the energy distribution of the cosmic gas of neutrinos left over from the big bang. Problem 2 investigates several applications of BBN as a cosmological probe. Problem 3 considers a cosmological model with anisotropic expansion. Problem 4 explores several applications of the freezeout calculation. Problem 5 asks you to think about why the Planck mass gives the energy scale at which quantum-gravity effects should become important. And Problem 6 works out chemical freezeout for the CMB and provides the scientific background for part of the motivation for this year's Nobel prize. Problem 2 should be given high priority and then problems 4 and 1. But problem 4 is one of my all-time favorites, and problem 6 (which is a bit more involved) is also quite interesting and might be fun.

1. **Cosmological relic neutrinos:** Suppose neutrinos have a mass of 1 eV. Then when their interactions freeze out in the early Universe at a temperature $T \simeq$MeV, they are relativistic, but today they are nonrelativistic. Find the energy distribution $f(K)$ (where K is the kinetic energy) of these cosmological neutrinos today.

2. **Big-bang nucleosynthesis as a cosmological probe:** As discussed in class, BBN can be used as a probe of possible deviations from the standard cosmological model and the standard model of particle interactions. The purpose of this problem is for you to work through some of these constraints.
   a. Explain how the primordial ^{4}He abundance would change if the neutron lifetime were longer or shorter.
   b. Suppose some particle theorists speculate that Fermi's constant G_F might actually be a function of time. What does BBN constrain the value of Fermi's constant to be at the time of nucleosynthesis?
   c. It is plausible that there is a neutrino-antineutrino asymmetry in the Universe, and therefore that the cosmological neutrino mass density *today* is greater than it is in the canonical picture. What is the upper limit to the current neutrino density Ω_ν provided by BBN (assume the neutrinos are massless).
   d. *Extra credit:* BBN also provides a constraint to the amplitude of the stochastic gravitational-wave background. Explain how this works, and the gravitational-wave frequency range to which it applies (Hint: Find a recent paper by the TA.)

3. **Anisotropic expansion:** Consider a Universe that undergoes anisotropic expansion. Such a Universe has a metric

$$ds^2 = dt^2 - a_x^2(t)\, dx^2 \;+\; a_y^2(t)\, dy^2 \;+ a_z^2(t)\, dz^2,$$

where $a_i(t)$ are scale factors of the three principal axes of the Universe.

a. Show that the Einstein equations for this metric lead to an analog of the Friedmann equation:

$$H^2 \equiv \frac{1}{9}\left(\frac{\dot{V}}{V}\right)^2 = \left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 = \frac{8\pi}{3m_{Pl}^2}(\rho + \rho_s),$$

where ρ is the ordinary energy density (radiation and nonrelativistic matter), and the shear "energy density" is defined to be

$$\rho_s \equiv \frac{m_{Pl}^2}{48\pi}[(H_x - H_y)^2 + (H_y - H_z)^2 + (H_x - H_z)^2].$$

Here $V = a_x a_y a_z$ is the "volume scale factor," $\bar{a} = V^{1/3}$ is the mean-scale factor, and the $H_i \equiv (\dot{a}_i/a_i)$ are the expansion rates of the three principal axes.

b. Next, show that the other Einstein equations (for $i \neq j$) become

$$\frac{d}{dt}\ln|H_i - H_j| = -3H = -3\frac{d}{dt}(\ln \bar{a}),$$

and that this implies that $\rho_s \propto \bar{a}^{-6}$. In other words, the effects of anisotropic expansion are mimicked by a new form of matter with energy density which decreases as $\bar{a}^{-6}$.

c. If ρ_s is too large, it will affect the results of BBN. Find the upper limit to ρ_s *today* provided by BBN.

d. Estimate how this BBN constraint to ρ_s might compare with the CMB constraint that $\Delta T/T < 10^{-5}$. Also, estimate how the BBN constraint might compare with ordinary measurements of the Hubble constant.

4. **Freezeout:**

a. Calculate the photon mean-free path and compare it with the Hubble distance H^{-1} at a redshift $z = 1500$, just before recombination.

b. After recombination, there will be a few residual electrons, due to freezeout of the recombination reaction $e^- + p \rightarrow H$. Calculate approximately the residual free-electron fraction. Make your assumptions about the capture cross section clear, and perhaps show how your result scales with that cross section.

c. In class we showed that a baryon-symmetric Universe would have a relic baryon density 10^9 times smaller than the observed density, from which we infer that there must be a cosmological baryon asymmetry characterized by a baryon-to-photon ratio today of $\eta = 5 \times 10^{-10}$. Calclate approximately the expected number of antibaryons in the entire observable Universe today, given this value for the baryon-to-photon asymmetry. Assume that the Universe remains perfectly homogeneous throughout its history. You may make reasonable approximations to get the answer correct to an order of magnitude or so.

5. **The Planck mass:** In class I claimed that at a temperature $T \simeq 10^{19}$ GeV, classical general relativity will break down, and quantum-gravity will become important. Justify this with at least one simple (but correct) physical argument.

6. **Chemical freezeout for photons and Nobel prizes for Mather and Smoot:** In class we showed that the rate $\Gamma_{\rm el}$ for photons to scatter elastically from electrons becomes less than the expansion rate H at a redshift of $z \simeq 1100$; in other words, $\Gamma_{\rm el}(z) = H(z)$ at $z = z_{\rm el} \simeq 1100$. At this redshift, photons are said to become *kinetically* decoupled from the plasma. Your job in this problem is to show that the rate for photon-number–changing interactions, such as the bremsstrahlung (or "free-free") reaction $e^- + p \leftrightarrow e^- + p + \gamma$, drops below the expansion rate at a much earlier time. To do so, *estimate* the rate for the bremsstrahlung reaction. You can either find the cross section in a book, or approximate it with a Feynmann diagram, if you are so inclined. Assume the electrons, protons, and photons involved all have energies comparable to the thermal temperature T. You should be able to show that the rate Γ for these reactions is bigger than the expansion rate H at early times, and then becomes comparable at a redshift $z = z_\mu \simeq 10^5 - 10^6$. After that, the rate for photon-number–changing interactions becomes smaller than H, and the photon number freezes out. What this means is that if someone (e.g., a long-lived decaying particle) were to inject photons into the primordial plasma at redshifts $z_\mu > z > z_{\rm el}$, those photons would come into *kinetic* equilibrium, but not *chemical* equilibrium. As a result, the frequency spectrum of the cosmic microwave background would be that of a Bose-Einstein gas with nonzero chemical potential μ. Measurements from the FIRAS experiment on NASA's COBE satellite constrain $\mu \lesssim 10^{-4}$; this measurement (and one other) was recognized by the 2006 Nobel prize for physics.