General Relativity Ph236c
Problem Set 4
Due: In class, May 1, 2007

Suggested Reading: Dodelson 6.4–6.6; Liddle and Lyth, *Cosmological Inflation and Large-Scale Structure*, Ch. 7; Carroll, 8.8; Kolb and Turner, Ch. 8; Peebles, Ch. 17.

1. **Direct detection of inflationary gravitational waves.** Using the formulas derived in class, calculate the rms gravitational-wave (GW) amplitude in the inflationary GW background in the LIGO/LISA frequency band (ask a friend or one of the many resident LIGO/LISA scientists or relativists here if you can't find the relevant frequencies rapidly in the literature). An order-of-magnitude calculation should be enough.

2. **Perturbations for $\lambda\phi^4$ inflation.** In a previous problem set, you considered inflation with a potential $V(\phi) = \lambda\phi^4$. You considered the homogeneous evolution of the scalar field, and the value of the scalar field at the end of inflation, and the number of e-foldings from some initial value ϕ_i. (a) Calculate the value of ϕ 60 e-folds before the end of inflation. This is presumably when the current observable horizon scale exits the horizon during inflation. Determine the values of the scalar and tensor spectral indices, n_s and n_t at this time. Estimate the curvature perturbation $\mathcal{R}$ at this time in terms of λ. What value of λ is required to get the right $\mathcal{R}$? What does this value of λ imply for the gravitational-wave amplitude h_k at the scale of the horizon today? To the best of your knowledge, is this model consistent with observations?

3. **Constancy of superhorizon curvature.** Using the relevant Einstein, continuity, and Euler-Lagrange equations, show that the curvature perturbation $\mathcal{R}$ remains constant when a given Fourier mode is well outside the horizon. Dodelson's book may be particularly useful for this problem. In fact, he more or less works it out, although in a different notation. If you can't at first work the problem out for yourself (it's not easy), then it may still be a good exercise to go through Dodelson's derivation and translate it into the notation and gauge choices we have used in class.

4. **Conformally coupled scalar field.** The Lagrangian density, for a noninteracting massive scalar field $\phi(x)$, that we have been using is

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left\{-g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2\right\}.$$

Such a scalar field is said to be *minimally coupled*, implying no coupling to gravity beyond replacing the Minkowski metric $\eta_{\mu\nu}$ with a more general metric $g_{\mu\nu}$. More generally, the Lagrangian can be written,

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left\{-g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - [m^2\phi^2 + \xi R\phi^2]\right\},$$

where $R(x)$ is the Ricci scalar, and ξ is a numerical factor.

a. Show that if $\xi = \frac{1}{4}[(n-2)/(n-1)]$ for n spacetime dimensions (e.g., $\xi = 1/6$ for $3+1$ spacetime dimensions), then the scalar-field action (and hence the field equations) is invariant under a conformal transformation,

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}(x),$$

where $\Omega(x)$ is a conformal factor. In this case, the scalar field is said to be *conformally coupled.*

b. Argue that there is no quantum production of scalar-field perturbations during inflation if the field is conformally coupled.

5. **No magnetic fields from inflation.** We saw in class that quantum effects during inflation give rise to classical superhorizon perturbations in a massless scalar field. The electromagnetic field is also a massless scalar field. Show, however, that there are no excitations of the electromagnetic field during inflation (it has to do with conformal invariance of the electromagnetic Lagrangian). Why do we demand conformal invariance of the electromagnetic Lagrangian? And why do you still get electromagnetic Hawking radiation from a black hole?

6. **Gauge degrees of freedom in linear theory.** We showed in quarter 1 that in linear perturbation theory (about a Minkowski spacetime), the trace-reversed metric perturbation $\bar{h}_{\alpha\beta}$ satisifies an equation,

$$\partial_\gamma \partial^\gamma \bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta},$$

(where $T_{\alpha\beta}$ is the stress-energy tensor) that looks a lot like Maxwell's equation ($\partial_\gamma \partial^\gamma A_\mu = -4\pi j_\mu$). In a vacuum, the the $\alpha 0$ component of this linearized equation becomes,

$$\partial_\gamma \partial^\gamma \bar{h}_{\alpha 0} = 0,$$

which admits wavelike solutions. Show that these wavelike solutions are pure gauge modes; i.e., they do not correspond to physically propagating waves. Then show that the wavelike solutions to the vacuum Maxwell equations can*not* be gauged away.