General Relativity (Ph236a)
Problem Set 5
Due: October 31, 2006

Preview: Problem 1 is a straightforward exercise to help develop your facility with tensors, metric, geodesics, etc., a good and relevant math exercise. Likewise, Problem 2 is intended to have you work through some steps involving gauge conditions in linear theory. Problem 3 is a really cute, and physically interesting, problem I lifted from Hartle's book. Problem 4 is a neat problem that investigates a possible non-minimal coupling of electromagnetism to gravity. Problem 5 has you work out the Newtonian limit of the stress-energy tensor; its a straightforward but good exercise. Problem 6 will have you work through an alternative-gravity theory. Problems 1–2 should be simpler than 3–6, but I would try to make 3–6 higher priority if you have limited time.

1. (From Lee Lindblom) **More practice with simple geometries and geodesics:** Consider the set of points that lie on the two-dimensional sphere, $r^2 = x^2 + y^2 + z^2$, where r is a constant.
   a. Argue that the projection operator $P_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the natural metric for this sphere, where $g_{\mu\nu}$ is the metric of the three-dimensional Euclidean space, and n_μ is the unit normal to the sphere.
   b. Show that coordinates can be chosen on the two-sphere so that the metric has the form $ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$.
   c. Write out the geodesic equations for the two-sphere geometry.
   d. Verify that the solutions to these equations correspond to the "great circles" on the sphere.

2. (Carroll, problem 7.4) **Gauge conditions:** Show that the Lorenz gauge condition $\partial_\mu \bar{h}^{\mu\nu} = 0$ is equivalent to the *harmonic gauge* condition, defined by

$$\Box x^\mu = 0,$$

   where each coordinate x^μ is thought of as a scalar function on spacetime. (Any function satisfying $\Box f = 0$ is known as an "harmonic function.")

3. (Hartle's problem 22.14) **The Alcubierre spacetime:** Consider the spacetime

$$ds^2 = -dt^2 + [dx - V_s(t) f(r_s) dt]^2 + dy^2 + dz^2,$$

   is specified by a curve $x_s(t)$, and $V_s(t) = dx_s(t)/dt$. The function $f(r_s)$ is any smooth positive function that satisfies $f(0) = 1$ and decreases away from the origin to vanish for $r_s > R$ for some R. Note that the spatial metric for any $t =$constant hypersurface is $dS^2 = dx^2 + dy^2 + dz^2$; i.e., the geometry of each spatial slice is flat and r_s is just the Euclidean distance from the curve $x_s(t)$. Inspection of the metric shows that spacetime is flat where $f(r_s)$ vanishes, but, as you will see below, it is curved where $f(r_s) \neq 0$.

a. Show that the local light cones are tipped in such a way that a spaceship moving along the curve $x_s(t)$ moves along a world line that is inside the local light cone, even if $V_s(t) > 1$. What this means is that the spaceship can travel between two stations separated by a distance D in a time (as viewed by a fixed external viewer) $T < D$.
b. Calculate the components n_α of the normal to a surface of constant t.
c. Use the Einstein equation to show that

$$T_{\alpha\beta} n^\alpha n^\beta = -\frac{1}{8\pi} \frac{V_s^2 (y^2 + z^2)}{(2r_s)^2} \left(\frac{df}{dr_s} \right)^2 .$$

This is the energy density measured by observers at rest with respect to the surfaces of constant t. The fact that it is negative means that this spacetime, which allows "superluminal" travel, cannot be supported by classical matter with a positive energy density.

4. (From Lee Lindblom) **Non-minimal gravitational coupling to electromagnetism:**
a. It has been suggested that quantum gravity might induce a curvature coupling to electromagnetism of the following form:

$$\nabla_\nu \left[(1 + \alpha R) F^{\mu\nu} \right] = 4\pi J^\mu, \qquad \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0,$$

where α is a constant and $R = R^\mu_{\ \mu}$ is the scalar curvature of spacetime. These equations reduce to the familiar Maxwell equations in flat spacetime (since $R = 0$ there). The fact that the second of these equations is unmodified means that we can still write $F_{\mu\nu}$ in terms of a vector potential $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, and the theory is still gauge invariant. Show that this version of the Maxwell equations, like the more conventional version, implies charge conservation. [*Hint*: First show that the antisymmetry of $F_{\mu\nu}$ implies $\nabla_\mu \nabla_\nu \left[(1 + \alpha R) F^{\mu\nu} \right] = (1 + \alpha R) \nabla_\mu \nabla_\nu F^{\mu\nu}$, then use that antisymmetry to show that this expression entails commutation of covariant derivatives, and show that the curvature terms produced by that commutation give a vanishing result.]
b. Quantum gravity is believed to introduce curvature couplings into the laws of physics, with coupling constants that involve the Planck length $l_{\rm Pl}$, the only combination of G, $\hbar$, and c with units of length. Write an expression for $l_{\rm Pl}$ in terms of these three fundamental constants and evaluate it in cgs units. By dimensional considerations, estimate the coupling constant α that quantum gravity might induce.
c. Perform a 3+1 split of these modified Maxwell equations in the local Lorentz frame of some observer; i.e., rewrite them in terms of the electric and magnetic fields $\vec{E}$ and $\vec{B}$ and the charge and current densities ρ and $\vec{j}$ that the observer measures. Discuss the physical manifestations of the curvature coupling that these 3+1 equations predict. Pay attention to the fact that R vanishes in a vacuum and that at the surface of some solid body, ∇R will have a delta-function behavior. You might want to consider, for example, Gauss' law, which usually expresses the total charge inside a body as a surface integral of the electric field emerging from the body. Estimate the dimensionless magnitude of these physical effects for experiments on Earth or in the solar system.

5. (MTW, problem 16.1) **Newtonian limit for fluids:**
   a. Consider a nearly Newtonian perfect fluid with stress-energy tensor,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu},$$

   with $p \ll \rho$, which moves with ordinary velocity $v^i = dx^i/dt \ll 1$ in a Newtonian spacetime,

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2).$$

   Show that the equations $\nabla_\mu T^{\mu\nu}$ reduce to the familiar equations for a Newtonian fluid moving in a gravitational field:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0,$$
$$\rho[\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v}] = -\vec{\nabla} p - \rho \vec{\nabla}\Phi.$$

6. (From Lee Lindblom) **Nordstrom's theory of gravity:** Consider spacetime geometries having the form $g_{\mu\nu} = \phi^2 \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat metric of special relativity.
   a. Show that the field equations $R = 24\pi T$, where $T = g^{\mu\nu}T_{\mu\nu}$ is equivalent to the equation $\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = -4\pi\phi\eta^{\mu\nu}T_{\mu\nu}$.
   b. Determine whether the Newtonian limit of this theory agrees with Newton's theory of gravity. That is, take the weak-field slow-motion limit of these equations and show whether they reduce to Newton's equations for the gravitational field. Also, determine whether the slow-motion geodesic equation reduces to the Newtonian equation for the motion of a particle in an external gravitational field.