General Relativity Ph236b
Problem Set 5
Due: In class, February 20, 2007

Preview: Problem 1 should be a pretty straightforward exercise in manipulation of Lagrangians, etc. Problem 2 represents a first step in the study of a parity-violating extension of electromagnetism that has been studied recently in the literature. Problem 3 is particularly interesting. It develops an alternative way to derive general relativity from the Einstein-Hilbert action, treating the connection as a quantity to be varied independently of the metric. This "Palatini formalism" has received considerable attention in the recent literature, as it leads to different gravity theories than the usual variation (the "metric formalism") for any action other than the Einstein-Hilbert action. Problem 4 is my favorite problem here. It is a fairly complicated problem, but should follow the derivation of the PPN parameter γ we did in class, but now for Brans-Dicke theory, instead of $1/R$ gravity. Problem 5 is an exercise to verify that the Schwarzchild metric is indeed the spherically-symmetric vacuum spacetime that extremizes the Einstein-Hilbert action. Problem 6 is a cute exercise that has you work out the gravitational field due to a point mass with a small extra dimension; it is relevant for experimental tests of gravity at small distances.

1. (Carroll, problem 1.12) **Electromagnetic energy-momentum tensor:** Consider electromagnetism (with $J^\mu = 0$) and scalar field theory, with Lagrangian $\mathcal{L} = -(1/2)\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$, both in flat spacetime.
   a. Express the components of the energy-momentum tensors of each theory in three-vector notation, using the divergence, gradient, curl, electric and magnetic fields, and an overdot to denote time derivatives.
   b. Using the equations of motion, verify (in any notation you like) that the energy-momentum tensors are conserved.

2. (Carroll, problem 1.13) **Chern-Simons electromagnetism:** Consider adding to the electromagnetic Lagrangian an additional term of the form $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$.
   a. Express $\mathcal{L}'$ in terms of $\vec{E}$ and $\vec{B}$.
   b. Show that including $\mathcal{L}'$ does not affect Maxwell's equations. Can you think of a deep reason for this?

3. (Carroll, problem 4.2) **Palatini formalism:** We showed how to derive Einstein's equation by varying the Hilbert action with respect to the metric. They can also be derived by treating the metric and connection as independent degrees of freedom and varying separately with respect to them; this is known as the *Palatini formalism.* That is, we consider the action,

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma),$$

   where the Ricci tensor is thought of as constructed purely from the connection, without reference to the metric. Variation with respect to the metric gives the usual Einstein equations, but for a Ricci tensor constructed from a connection that has no a priori relationship to the metric. Imagining from teh start that the connection is symmetric (torsion free), show that variation of the action with respect to the connection coefficients leads to the requirement that the connection be metric compatible, that is, the Christoffel connection. Remember that Stokes' theorem, relating the integral of the covariant divergence of a vector to an integral of the vector over the boundary, does not work for a general covariant derivative. The best strategy is to write the connection coefficients as a sum of the Christoffel symbols $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ and a tensor $C^{\lambda}_{\mu\nu}$,

$$\Gamma^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} + C^{\lambda}_{\mu\nu},$$

and then show that $C^{\lambda}_{\mu\nu}$ must vanish. Although the Palatini formalism gives the same physics as the usual technique (the "metric formalism") for the Einstein-Hilbert action, the two formalisms do not necessarily yield the same equations of motion or connection for more complicated actions. Thus, when considering alternatives to GR, one must specify whether one is dealing with the Palatini or the metric formalism, in addition to the action.

4. **PPN parameter for Brans-Dicke theory:** In this problem you will calculate the parameterized post-Newtonian (PPN) parameter γ in Brans-Dicke theory in terms of the Brans-Dicke parameter ω, and you will also relate the Newtonian gravitational constant to ω and the value of the Brans-Dicke scalar λ (in Carroll's notation). To do so, consider the weak-field limit of the spherically-symmetric static vacuum spacetime that surrounds a spherical massive body (e.g., the Sun). Thus, write the spacetime metric as

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\,d\Omega^2,$$

with $A(r) = 1 + a(r)$ and $B(r) = 1 + b(r)$, where $a(r) \ll 1$ and $b(r) \ll 1$ (and are both linear in the central mass M) in the weak-field limit. You will also need to write the Brans-Dicke scalar as $\lambda = \lambda_0[1 + \epsilon(r)]$, where λ_0 is the value of the Brans-Dicke scalar at large distances, and $\epsilon(r) \ll 1$ (and will also be linear in M). You will then obtain equations of motion for $a(r)$, $b(r)$, and $\epsilon(r)$ by plugging into the Brans-Dicke field equations (for the metric and for the scalar field) retaining terms only to linear order in $a(r)$, $b(r)$, and $\epsilon(r)$. The Newtonian limit is taken by identifying $a(r) = 2GM/r$ (do you know/remember why? if not, think about it), and the PPN parameter γ is defined from $b(r) = 2\gamma GM/r$.

5. **The Schwarzchild spacetime and variational principle:** Show explicitly that the spherically symmetric static vacuum spacetime that minimizes the Einstein-Hilbert action is the Schwarzchild metric. To do so, write the metric in the form,

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\,d\Omega^2,$$

and show explicitly that the Einstein-Hilbert action is invariant to any linear variations to $A(r) = (1 - 2M/r)$ and $B(r) = (1 - 2M/r)^{-1}$.

6. (Problem 3.9 in Zwiebach's "A first course in string theory") **Gravitational field of a point mass in a compactified (4+1)-dimensional world:** In this problem you

will show that if there is an extra dimension, curled up into a circle of radius a, then at distanves $r \gg a$, the gravitational force due to a point mass decreases as $1/r^2$, but that at distances $r \ll a$, the force law rises (as $r \to 0$) as $1/r^3$. This departure, at small distance scales, from the $1/r^2$ force law has now been sought in the laboratory (most notably by Eric Adelberger and collaborators in Seattle) at distances smaller than 1mm. OK, Here's the problem: Consider a $(4+1)$-d spacetime with space coordinates (x, y, z, w) *not* yet compactified. A point mass M is located at the origin $(x, y, z, w) = (0, 0, 0, 0)$.

a. Find the gravitational potential $V_g^{(5)}(r)$. Write your answer in terms of M, the five-dimensional gravitational constant G_5, and $r = (x^2 + y^2 + z^2 + w^2)^{1/2}$. [Hint: Use $\nabla^2 V_g^{(5)} = 4\pi G_5 \rho_m$ and the divergence theorem.]
b. Now let w become a compact dimension, a circle with radius a, while keeping the mass fixed. Write an exact expression for the gravitational potential $V_g^{(5)}(x, y, z, 0)$. This potential is a function of $R \equiv (x^2 + y^2 + z^2)^{1/2}$ and can be written as an infinite sum.
c. Show that for $R \gg a$, the gravitational potential takes the form of a four-dimensional gravitational potential, with Newton's constant G_4 given in terms of G_5, as derived in class. [Hint: Turn the infinite sum into an integral.]