General Relativity (Ph236a)
Problem Set 6
Due: November 7, 2006

Preview: Problem 1 is a great problem; it is an excellent exercise in the first post-Newtonian correction of general relativity, and it works out the Lense-Thirring effect and sheds light on Mach's principle in general relativity. Problem 2 uses essentially the same physics as problem 1; its a cute problem, but I would make it lower priority if you have limited time. Problem 3 is a cool problem, showing that photon trajectories in linear theory can be determined from Fermat's principle—still, maybe lower priority if you have limited time. Problem 4 is a real problem in astrophysics; those of you interested in astrophysics should make this one high priority. Similarly for problem 5. Problem 6 gets into the nitty gritty of gravitational lensing and can be considered lower priority.

1. (Carroll 7.2; Wald 4.3) **Lense-Thirring effect:** Consider a thin spherical shell of matter with mass M and radius R, slowly rotating (i.e., maximum rotation velocity $v = \Omega R \ll c$) with an angular velocity Ω.
   a. Show that the gravito-magnetic field $\vec{G}$ vanishes and calculate the gravitomagnetic field $\vec{H}$ in terms of M, R, and Ω.
   b. The nonzero gravitomagnetic field caused by the shell leads to dragging of inertial frames, known as the *Lense-Thirring effect*. Calculate the rotation (relative the inertial frame defined by the background Minkowski metric) of a freely-falling observer sitting at the center of the shell. In other words, calculate the precession of the spatial components of a parallel-transported vector located at the center.

2. **Gravitomagnetism in the Solar System:** Suppose a spherical body of uniform density ρ and radius R rotates rigidly about the x^3 axis with constant angular velocity Ω. (a) Write down the components $T^{0\nu}$ of the stress tensor to lowest order in ΩR in a Lorentz frame at rest with respect to the center of mass of the body, assuming constant ρ, Ω, and R. (b) Calculate $\bar{h}^{00}$ and $\bar{h}^{0j}$ outside the body to lower nonvanishing order in r^{-1} (where r is the radius). Express the result in terms of the body's angular momentum, and write the metric tensor in this approximation in spherical coordinates. (c) Since the metric is independent of time t and azimuthal angle ϕ, the components of the energy-momentum p_0 and p_ϕ will be constant along the trajectory of orbiting particles. Consider a particle of nonzero rest mass in a circular orbit of radius r in the equatorial plane. Calculate (to lowest order) the difference between the orbital period of a particle in a co-rotating orbit and that of a particle in a counter-rotating orbit. (Here the period is the coordinate time taken for one orbit $\Delta\phi = 2\pi$. (d) How much shorter or longer would the Earth's year be if the Earth revolved about the Sun in the opposite direction? ($M_\odot = 2 \times 10^{30}$ kg, $R_\odot = 7 \times 10^8$ m, $\Omega = 3 \times 10^{-6}$ sec^{-1}, and $r = 1.5 \times 10^{11}$ m).

3. (Carroll, problem 7.3) **Fermat's principle and photon trajectories:** Fermat's principle states that a light ray moves along a path of least time. For a medium with refractive

index $n(\vec{x})$, this is equivalent to extremizing the time,

$$t = \int n(\vec{x}) \, [\delta_{ij} \, dx^i \, dx^j]^{1/2},$$

along the path. Show that Fermat's principle, with the refractive index given by $n = 1 - 2\Phi$, leads to the correct equation of motion for a photon in a spacetime perturbed by a Newtonian potential.

4. **Gravitational microlensing in the Milky Way halo:** Dynamical measurements indicate that the disk and bulge of the Milky Way are immersed in a dark halo (assumed to be—and likely close to—spherically symmetric) with a mass-density distribution,

$$\rho(r) = \rho_0 \frac{r_0^2 + a^2}{r^2 + a^2},$$

where r is the distance from the Galactic center, $\rho_0 = \rho(r_0) \simeq 0.4$ GeV/cm^3 is the local halo density, $r_0 \simeq 8.5$ kpc is our distance from the Galactic center, and $a \simeq 4$ kpc is the core radius. (Note that 1 kpc= 1000 pc, and pc=3×10^{18} cm.) Now suppose that this dark halo was composed of neutron stars, each of mass $1.4\, M_\odot$, where $M_\odot = 2 \times 10^{33}$ erg is the mass of the Sun. Suppose further that we look at a star in the Large Magellanic Cloud, a dwarf galaxy located at a distance 50 kpc from us, and we'll assume here that it is close to the Southern Galactic pole. If a halo neutron star passes near the line of sight to one of the stars in the Large Magellanic Cloud, then that star will be gravitationally lensed.

   a. Estimate the Einstein radius (in arcsec) of the lens; you should find that this is very small compared with the resolution ($\sim$arcsec) of ground-based optical telescopes. If so, then the two images that appear when the source is strongly lensed will be unresolved.

   b. Estimate the physical Einstein radius and compare it with the Schwarzchild radius. Does this result justify our use of the weak-field limit?

   c. The "optical depth" for microlensing is the probability that the line of sight to a given star in the LMC falls within the Einstein ring of one the dark-halo neutron stars. Estimate this optical depth. The inverse of this optical depth is the number of stars in the LMC you would need to monitor if, on average, one star is strongly lensed at any given time.

   d. Calculate the magnification of the two images during lensing, and plot it as a function of the impact parameter.

   e. Of course, the lenses in the Galactic halo are moving, and so a source will not remain within the Einstein radius for more than a fixed amount of time. Estimate the time that a source spends within in the Einstein radius during a typical microlensing event.

   f. Calculate and plot the brightness of a background star, as a function of time, as it passes behind the lens, for a minimum impact parameter b. Assume the relative angular velocity of lens and source is constant.

5. **Gravitational lensing of quasars:** Quasars are extraordinarily bright cosmological sources fueled by accretion onto supermassive black holes at the centers of very distant

galaxies. They are sometimes seen in multiple images, a result of gravitational lensing by lenses along the line of sight. These lenses are usually elliptical galaxies.

a. Suppose the orbital speeds $v_c(r)$ of stars on circular orbits in the elliptical galaxy remains constant as a function of distance r from the elliptical-galaxy center. Show that this imples a mass-density profile $\rho(r) \propto r^{-2}$. (A typical rotation speed is $v_c \simeq 300$ km/sec.)
b. Calculate the deflection angle for lensing by this elliptical galaxy as a function of the circular speed v_c.
b. Estimate (order of magnitude) the characteristic separation (be sure your result makes sense) of the sources, and then estimate (again, order of magnitude) the characteristic time delay between different sources. Assume the quasars are at a distance of a Gpc ($= 10^9$ pc).

6. **Source rotation by gravitational lensing:** In class we wrote down a magnification tensor A for gravitational lensing. This tensor described the mapping from the image plane to the source plane (or *vice versa*), and it consisted of a convergence κ and two components, γ_1 and γ_2, of the shear. How would this tensor need to be modified if lensing *rotated* a source by an angle α? Is it possible for lensing to introduce a rotation?