General Relativity Ph236b
Problem Set 6
Due: In class, February 27, 2007

Preview: Problem 1 asks you to fill in some steps in the definition of the surface gravity; it should be straightforward. Problem 2 deals with the Kerr spacetime; its a good physics problem. Problem 3 is an interesting mathematical result involving extreme Reissner-Nordstrom black holes. Problem 4 has you calculate the surface gravity of the horizon in de Sitter space. Problem 5 should be a very simple algebraic exercise. Problem 6 has you think more deeply about conformal diagrams; this is pretty sophisticated stuff, so you should be proud if you can figure it out.

1. **Surface gravity for a Killing horizon:** Consider a Killing vector χ^μ with Killing horizon Σ. Along Σ, χ^μ satisfies the geodesic equation $\chi^\mu \nabla_\mu \chi^\nu = -\kappa \chi^\mu$. Using Killing's equations $\nabla_{(\mu}\chi_{\nu(} = 0$ and the fact that $\chi_{[\mu}\nabla_\nu \chi_{\sigma]} = 0$ (as you showed in Problem 1(a) in Problem Set 9 of the first quarter, if χ^μ is orthogonal to Σ), show that

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi^\nu)(\nabla^\mu \chi_\nu),$$

[Equation (6.9) in Carroll's book].

2. **Proper distances in the nearly-maximally-rotating black hole:** In our discussion of the Kerr black hole in week 1 of this quarter, we considered several different radii: (i) the horizon radius r_+, (ii) the limiting-circular-photon-orbit radius $r_{\rm ph}$, (iii) the marginally-bound-orbit radius $r_{\rm mb}$, the marginally-stable-circular-orbit (or innermost stable circular orbit; ISCO) $r_{\rm ms}$. In the limit that the spin parameter $a \to M$, all of these radii approach the value M. It thus appears that the positions of all these points become co-located. However, since $g^{rr} \to 0$ in this limit, a small *coordinate* separation may represent a large *proper* separation. Thus, consider a Kerr black hole with spin parameter $a = M(1-\epsilon)$, with $\epsilon \ll 1$, and calculate, in this limit, the proper separation between these different radii in terms of M and ϵ.

3. (Carroll, problem 6.1) **Extreme Reissner-Nordstrom black holes:** Show that the coupled Einstein-Maxwell equations can be simultaneously solved by the metric,

$$ds^2 = -H^{-2}(\vec{x})dt^2 + H^2(\vec{x})[dx^2 + dy^2 + dz^2],$$

where $H = 1 + (M/|\vec{x}|)$, and the electrostatic potential,

$$A_0 = H^{-1} - 1,$$

if $H(\vec{x})$ obeys Laplace's equation, $\nabla^2 H = 0$.

4. (Carroll, problem 6.4) **Surface gravity of the de-Sitter-space horizon:** Consider de Sitter space in static coordinates:

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2 d\Omega^2.$$

This space has a Killing vector ∂_t that is timelike near $r = 0$ and null on a Killing horizon. Locate the radial position r_K of the Killing horizon. What is the surface gravity κ of the Killing horizon?

5. **de Sitter space:** Find the coordinate transformation that takes the de Sitter space metric

$$ds^2 = -(dt')^2 + \alpha^2 \cosh^2(t'/\alpha)[d\chi^2 + \sin^2\chi d\Omega^2],$$

to the more familiar form,

$$ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2 d\Omega^2);$$

be sure to state how H is related to α. Show from this coordinate transformation that the latter metric does not cover the complete manifold [cf., Problem 6(b) of Problem Set 2 this quarter].

6. **Conformal diagrams for black holes with a cosmological constant:** Two-dimensional slices of a static metric can be written in the form,

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)}.$$

a. The Schwarzchild-de-Sitter spacetime has

$$F = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2,$$

(as you showed in Problem 6 of Problem Set 8 in the first quarter). Take $\Lambda > 0$ and draw the conformal diagram for this spacetime for (i) $3M < \Lambda^{-1/2}$, (ii) $3M = \Lambda^{-1/2}$, and (iii) $3M > \Lambda^{-1/2}$.

b. Draw the conformal diagram for the Schwarzchild-anti-de-Sitter spacetime, given by F above with $\Lambda < 0$.

c. If you're *really* ambitious, you can also try drawing the conformal diagram for the Reissner-Nordstrom-de-Sitter spacetime, which has

$$F = 1 - \frac{2M}{r} - \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2.$$

Note: This may be a hard problem. If you have trouble, you may want to look at "Effects of a nonvanishing cosmological constant on the spherically symmetric vacuum manifold," K. Lake and R. C. Roeder, *Phys. Rev. D* **15**, 3513 (1977).