General Relativity, Ph236c, Spring 2007

Problem Set 6

Due: May 29, 2007

1. Consider a critical-density Universe in which massive neutrinos contribute Ω_ν to the density parameter. Show that on scales smaller than the neutrino Jeans length, perturbations in the remaining cold component grow as $\delta \propto t^\alpha$, where $\alpha = (\sqrt{25 - 24\Omega_\nu} - 1)/6$. (Hint: The Ω_ν of the critical density in neutrinos contributes to the expansion rate, but this component remains smoothly distributed.)

2. In this problem you will explore numerically the growth of a spherical perturbation in a cosmological-constant Universe. A spherically-symmetric perturbation collapses in a flat cosmological-constant Universe (i.e., $\Omega_m + \Omega_\Lambda = 1$) of arbitrary Ω_m. Derive an exact density contrast at virialization (you will probably not be able to do this analytically), and compare with the oft-quoted estimate, $1 + \delta = 178\Omega_m^{-0.7}$.

3. Consider linear growth of perturbations in a Universe with cold dark matter with density $\Omega_{\rm cdm} = 0.25$ and baryon density $\Omega_b = 0.05$. Consider only redshifts $z \gg 1$ so that the dynamical effect of the cosmological constant is negligible. Write down the differential equations for linear evolution of $\delta_{\rm cdm}(\vec{x}, t) = \delta\rho_{\rm cdm}(\vec{x}, t)/\bar{\rho}_{\rm cdm}$, the fractional perturbation to the CDM density, and for $\delta_b(\vec{x}, t) = \delta\rho_b(\vec{x}, t)/\bar{\rho}_b$, the fractional perturbation to the baryon density. Now consider the evolution of a single Fourier mode of wavelength λ and wavenumber $k = 2\pi/\lambda$, of the density field. Show that baryon perturbations are stabilized by pressure at small scales, and find an expression for the Jeans wavelength Λ_J, the wavelength that separates stable and unstable modes. Evaluate the Jeans wavelength just before and just after recombination, and determine the corresponding Jeans mass.

4. Calculate the neutrino damping length as a function of the neutrino mass m_ν (for values of m_ν between 0.1 eV and 1 eV). Assume that all three neutrino species have the same mass.

5. Calculate the relation between the one-dimensional power spectrum $P_{1D}(k)$ (as would be measured in a "pencil-beam" survey) the usual three-dimensional power spectrum $P(k)$.

6. Calculate (numerically) and plot the root-variance $\sigma(M)$ as a function of mass M for the ΛCDM power spectrum $P(k)$ for $\Omega_m = 0.3$. Then, determine M_* (defined by $\sigma(M_*, t) = \delta_c$ as a function of redshift from $z = 0$ to $z = 100$. Use the proper linear-theory growth factor (you can use the semi-analytic approximation given in class), and also the proper δ_c (also using the approximation given in class).

7. Suppose that the 1-point distribution function, normalized to unit variance is $\mathcal{P}(\nu)$ (e.g., for Gaussian initial conditions, $\mathcal{P}\nu = (2\pi)^{-1/2}\exp(-\nu^2/2)$). (a) Show that the usual Press-Schecter equation for the number of gravitationally-bound halos with masses

between M and $M + dM$ per comoving volume at redshift z generalizes to

$$\frac{dn}{dM}dM = \frac{f\rho_b}{M}\mathcal{P}[\nu(M,z)]\frac{\partial\nu(M,z)}{\partial M}dM,$$

where $\nu = \delta_c(z)/\sigma_M$, $\delta_c(z) = 1.69/D(z)$ is the critical overdensity for gravitational collapse, $D(z)$ is the linear-theory growth factor, and σ_M is the variance of the mass distribution for scales M. Also, $f = \int_0^\infty \mathcal{P}(\nu)\,d\nu$. (b) Now suppose that once halos form their mass is fixed, and suppose further that they disappear only when they merge into larger halos. Show that with these assumptions, the distribution (normalized to unity) of formation redshifts z_f for halos of mass M observed at redshift z_0 is

$$\frac{df}{dz_f} = \mathcal{P}'[\nu(M,z_f)]\frac{\partial\nu(M,z_f)}{\partial z_f}\left\{\mathcal{P}[\nu(M,z_0)]\right\}^{-1}.$$

(c) Evaluate this formation-redshift distribution for a Gaussian distribution of perturbations and describe it qualitatively. For hints, see *MNRAS* **321**, L7 (2001).