General Relativity (Ph236a)
Problem Set 7
Due: November 14, 2006

Preview: Problem 1 is an exercise to get you to think about the effects of gravitational waves on test masses. Problems 2 and 3 ask you to fill in steps in the derivation of the quadrupole formula—they can be considered lower priority if you have limited time. Problem 4 might be entertaining. Problem 5 is a standard calculation, relevant in particular for the Nobel prize that went to Hulse and Taylor for discovery of the binary pulsar—that prize was awarded largely because the observed frequency change of the Hulse-Taylor pulsar was in excellent agreement with the GR prediction, which you calculate in this problem. Gravitational-wave detection is now the aim of a vast experimental effort, and Problem 6 has you work through some numerical estimates relevant for experiments.

1. **Circular and elliptically polarized gravitational waves:**
   a. A gravitationl wave is said to be circularly polarized if a + polarized gravitational wave is added to a × polarized gravitational wave with the same amplitude and frequency, but 90° out of phase. Show that a circularly polarized gravitational wave with frequency ω that is normally incident on an ellipse of test masses causes each particle to rotate in a small circle such that the elliptical pattern rotates with a constant angular frequency. What is that angular frequency?
   b. A wave propagating in the $\hat{\mathbf{z}}$ direction is said to be elliptically polarized with principal axes x and y if $h^{\rm TT}_{xy} = iah^{\rm TT}_{xx}$, where a is some real number, and $h^{\rm TT}_{yy} = -h^{\rm TT}_{xx}$. Show that if $h^{\rm TT}_{xy} = \alpha h^{\rm TT}_{xx}$, where α is a complex number (the general case for a plane wave), new axes x' and y' can be found for which the wave is elliptically polarized with principal axes x' and y'.
   c. Describe what happens when an elliptically polarized gravitational wave is normally incident on a circle of test masses.
   d. Show that circular and linear polarization are special cases of elliptical polarization.

2. **The tensor virial theorem:** Show that in the nonrelativistic limit,

$$\frac{d^2}{dt^2}\int T^{00} x^l x^m \, d^3\vec{x} = 2\int T^{lm}\, d^3\vec{x},$$

   for a bounded system. This is the tensor virial theorem.

3. **Angular integrals:** If n^i are the components of an ordinary three-dimensional unit vector $\hat{\mathbf{n}}$, show that

$$\int n^j n^k \sin\theta \, d\theta \, d\phi = \frac{4\pi}{3}\delta_{jk},$$

   and

$$\int n^i n^j n^k n^l \, \sin\theta \, d\theta \, d\phi = \frac{4\pi}{15}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).$$

4. **Graviton emission:** The photon is the quantum of the electromagnetic field, and a photon of frequency ν has an energy $h\nu$, where h is Planck's constant. Likewise, the gravitational field has presumably (we haven't detected one yet) quanta that we refer to as "gravitons." Unlike the photon, which is spin 1, the graviton must be spin 2, but it should also have an energy $h\nu$. The TAPIR interaction room features an electrically powered "gravitational-wave generator." When turned on, it spins two masses around in a circle. I haven't done the measurements, but I'm guessing each mass is about 0.1 kg, the masses are separated by about 25 cm, and the frequency is probably around 5 rotations per second. Use the quadrupole formula to estimate the rate at which gravitons are emitted from the gravitational-wave generator.

5. (Wald 4.9) **Binary inspiral:** A binary star system consists of two stars of mass M and negligible size in a nearly Newtonian circular orbit of radius R around each other. Assuming the validity of the quadrupole approximation, calculate the rate of increase of the orbital frequency due to emission of gravitational radiation.

6. **Plugging in the numbers:**
   a. In 2003, a double pulsar (PSR J0737-3039), a binary pulsar consisting of two neutron stars (each of mass roughly 1.4 $M_{\odot}$, with an orbital period 2.4 hours), was discovered. Estimate the timescale for coalescence via emission of gravitational radiation.
   b. Suppose that two supermassive (let's say $10^9\,M_{\odot}$) black holes merge in a galaxy at a cosmological distance (let's say 1 Gpc). Consider a detector built to detect gravitational waves from such events. (i) Estimate the frequency range that the detector would have to operate at. (ii) Estimate the strain sensitivity (smallest dimensionless amplitude h) that would be necessary to see mergers at these distances. (iii) Estimate the duration of these events.
   c. The LIGO gravitational-wave detector expects to detect gravitational waves at frequencies $\sim$ 200 Hz with dimensionless strains of $\sim 10^{-21}$. (i) What is the flux of energy of such waves? (ii) If they come from 20 Mpc, what is the luminosity of the source? (iii) How far away would the Sun have to be to produce the same flux in electromagnetic radiation?