General Relativity Ph236b

Problem Set 7

Due: In class, March 6, 2007

Preview: The first five problems are all exercises on differential forms. Problem 1 is just an algebraic exercise. Problems 2 and 3 have you think about Maxwell's equations in the language of differential forms. Problem 4 is a neat problem that has you work through some aspects of gauge theories in higher dimensions. Problem 5 is a good exercise for you to understand how to calculate the Riemann tensor using the tetrad formalism. As a warmup, you may want to read through Ch. 6.1 on Wald's book to see, as an example, this formalism used to calculate the Riemann tensor for the Schwarzchild spacetime. Problem 6 is a cute problem (that really has nothing to do with GR) in Newtonian mechanics.

1. (Carroll, problem 2.8) **Leibnitz rule for the exterior derivative:** Show that the action of the exterior derivative d on the wedge product of a product of a p-form ω and a q-form η is

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta).$$

2. (Carroll, problem 2.9) **Euclidean-space E&M:** In Euclidean three-space, suppose $\star F = q \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$.
   a. Evaluate $\mathrm{d} \star F = \star J$.
   b. What is the two-form F equal to?
   c. What are the electric and magnetic fields equal to?
   d. Evalute $\int_V \mathrm{d} \star F$, where V is a ball of radius R in a Euclidean three-space.

3. (Carroll, problem 2.10) **(1+1)-d Maxwell's equations:** Consider Maxwell's equation, $\mathrm{d}F = 0$, $\mathrm{d} \star F = \star J$, in a 2-dimensional spacetime. Explain why one of the two sets of equations can be discarded. Show that the electromagnetic field can be expressed in terms of a scalar field. Write out the field equations for this scalar field in component form.

4. (Carroll, problem 2.11) **Extra-dimensional gauge theories:** This is problem 2.11 from Carroll's book. As you'll see, its a long problem, and I'm too lazy to type it in. You, however, should not be too lazy to solve it; its a good one.

5. (Carroll, problem J.2) **Tetrad formalism for the mixmaster universe:** Calculate the connection one-forms, curvature two-forms, and hence the components of the Riemann tensor for the Mixmaster universe. The metric is given by

$$ds^2 = -\mathrm{d}t \otimes \mathrm{d}t + \alpha^2 \sigma^1 \otimes \sigma^1 + \beta^2 \sigma^2 \otimes \sigma^2 + \gamma^2 \sigma^3 \otimes \sigma^3.$$

Here α, β, and γ are functions of t only, and the one-forms σ^i are given by

$$\begin{aligned}
\sigma^1 &= \cos\psi \mathrm{d}\theta + \sin\psi \sin\theta \mathrm{d}\phi, \\
\sigma^2 &= \sin\psi \mathrm{d}\theta - \cos\psi \sin\theta \mathrm{d}\phi, \\
\sigma^3 &= \mathrm{d}\psi + \cos\theta \mathrm{d}\phi.
\end{aligned}$$

6. **Extra-dimensional Keplerian motion:** The solution is supposed to show that if the Universe had more than three spatial dimensions, then the orbit of the Earth around the Sun would not be stable, and hence unable to support life. This argument has been given (perhaps tongue in cheek) as an anthropic argument for why the Universe has three spatial dimensions. In three spatial dimensions, the gravitational acceleration due to a point mass (e.g., the Sun in the Solar System) is proportional to $1/r^2$, where r is the distance from the point mass.
   a. Show that with this force law, circular orbits are stable to small perturbations.
   b. With d extra spatial dimensions, the force law becomes proportional to $1/r^{2+d}$. Determine the condition on d for circular orbits to be stable.