General Relativity (Ph236a)
Problem Set 8
Due: November 21, 2006

Preview: Problem 1 fills in some differential geometry that we don't have time to get to in class. Its interesting, but more mathematical than physical. Problems 2, 3, and 4 are exercises to help develop facility in dealing with Killing vectors; they are mathematical, but there is considerable connection with important physics. Problems 5 and 6 deal with interesting non-Schwarzchild spherically-symmetric spacetimes.

1. **Lie derivatives:** The *Lie derivative* is another derivative operator that acts in a curved manifold on an arbitrary tensor field. We will not discuss it in class, as it is somewhat outside the main line of the development of GR, but it does come in handy in formal developments and in other applications of GR. See Appendix C.2 in Wald or Appendix B in Carroll for formal definitions; Weinberg's Ch. 10.9 shows how the Lie derivative generates changes to tensors under coordinate (gauge) transformations. The *Lie bracket* for two vector fields $\mathbf{U}$ and $\mathbf{V}$ is defined to be the vector field $[\mathbf{U},\mathbf{V}]$ with components,

$$[\mathbf{U},\mathbf{V}]^\alpha = U^\beta \partial_\beta V^\alpha - V^\beta \partial_\beta U^\alpha;$$

i.e., this is the commutator that you've studied in problem sets before.

a. Show that $[\mathbf{U},\mathbf{V}]$ is a derivative operator on $\mathbf{V}$ along $\mathbf{U}$; i.e., show that for any scalar function f,

$$[\mathbf{U}, f\mathbf{V}] = f[\mathbf{U},\mathbf{V}] + \mathbf{V}(\mathbf{U}\cdot\nabla f).$$

The *Lie derivative* of $\mathbf{V}$ with respect to $\mathbf{U}$ is denoted by

$$[\mathbf{U},\mathbf{V}] \equiv \mathcal{L}_{\mathbf{U}}\mathbf{V},$$

and the action of the Lie derivative on a scalar function is

$$\mathbf{U}\cdot\nabla f \equiv \mathcal{L}_{\mathbf{U}} f.$$

You have thus verified that the Lie derivative satisfies (when acting on vectors and scalars) the Leibniz rule,

$$\mathcal{L}_{\mathbf{U}}(f\mathbf{V}) = f\mathcal{L}_{\mathbf{U}}\mathbf{V} + \mathbf{V}\mathcal{L}_{\mathbf{U}} f.$$

You may recall that the covariant derivative of a function is equal to the partial derivative, and also from your earlier homework on the commutator, that the commutator does not depend on the connection. The Lie derivative is therefore independent of the connection.

b. Calculate the components of the Lie derivative of a one-form field $\tilde{\omega}$ from the knowledge that, for any vector field $\mathbf{V}$, $\tilde{\omega}(\mathbf{V}$ is a scalar (like f above), and from the definition that $\mathcal{L}_{\mathbf{U}}\tilde{\omega}$ is a one-form field:

$$\mathcal{L}_{\mathbf{U}}[\tilde{\omega}(\mathbf{V})] = (\mathcal{L}_{\mathbf{U}}\tilde{\omega})(\mathbf{V}) + \tilde{\omega}(\mathcal{L}_{\mathbf{U}}\mathbf{V}).$$

c. Show that the Lie derivative satisfies

$$\mathcal{L}_{\mathbf{U}}\mathcal{L}_{\mathbf{V}} - \mathcal{L}_{\mathbf{V}}\mathcal{L}_{\mathbf{U}} = \mathcal{L}_{[\mathbf{U},\mathbf{V}]}.$$

d. A vector field $\mathbf{V}$ is said to be "Lie transported" (or "dragged") along a vector field $\mathbf{V}$ if $\mathcal{L}_{\mathbf{V}}\mathbf{U} = 0$. Discuss the differences between (i) parallel transport of a vector, (ii) Fermi-Walker transport, and (iii) Lie transport.

2. (from Lee Lindblom) **Killing vector fields:**
   a. Let $\xi^a\partial_a$ be an arbitrary vector field in an arbitrary curved spacetime. Show that it is possible to construct a coordinate system in which ξ^a is one of the coordinate basis vectors. That is, show that there are coordinates so that $\xi^a\partial_a = \partial/\partial x^K$, where x^K is one of these coordinates.
   b. Show that in this coordinate system,

$$\nabla_a\xi_b + \nabla_b\xi_a = \xi^c\partial_c g_{ab} = \partial g_{ab}/\partial x^K.$$

   Thus, in this coordinate system, ξ^a satisfies Killing's equation if and only if the metric components are independent of the coordinate x^K.
   c. Assume that xi^a and ζ^a are both Killing vector fields. Then show that their commutator $[\xi,\zeta]^a$ is also a Killing vector field, and that $\alpha\xi^a + \beta\zeta^a$ is a Killing vector when α and β are constants.

3. (from Lee) **Killing transport:**
   a. Show that $\nabla_a\nabla_b\xi_c = \xi_d R^d{}_{abc}$ for any Killing vector xi^a.
   b. If ξ_a is a Killing vector field, and $F_{ab} = \nabla_a\xi_b$, show that ξ_a and F_{ab} are transported along an arbitrary curve with tangent vector v^a according to the equations:

$$v^a\nabla_a\xi_b = F_{ab}v^a, \qquad v^a\nabla_a F_{bc} = R^d{}_{abc}\xi_d v^a.$$

   Show that there are a system of ordinary differential equations for ξ_a and F_{ab} along the curve.
   c. Determine the number of linearly independent initial conditions for these Killing transport equations at a given spacetime point. Show that there can be no more than $n(n+1)/2$ independent Killing vector fields in any space of dimension n.

4. (from Lee) **Killing vectors in flat spacetime:**
   a. Find conditions on the constant tensor F_{ab} so that $\xi_a = F_{ab}x^b$ is a Killing vector field in Minkowski space where x^a represent Cartesian coordinates.
   b. Show that the Killing vectors constructed in (a) correspond exactly to the rotations and the Lorentz boosts.
   c. Find four additional Killing vectors in Minkowski space no included in (a). Argue that these, together with those found in (a), are all the Killing vectors of flat spacetime.

5. (Hartle, problem 22.15) **Wormholes require negative energy density:** Consider the spacetime described by the metric,

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2).$$

Here, the coordinate $-\infty < r < \infty$. This metric describes a spacetime that consists of two asymptotically flat spacetimes (at $r \to \infty$ and $r \to -\infty$) connected by a throat at $r = 0$ of radius $2\pi b$; if its not clear, see Fig. 7.5 in Hartle's book. Calculate the components of $T^{\alpha\beta}$ that would be needed for this geometry to be a solution of the Einstein equation. Show that the energy density (as measured by a stationary observer) required is *negative*. Thus, until someone figures out a way to create negative-energy-density matter, we're not gonna be traveling through any such worhmholes.

6. (Carroll, problem 5.4) **Spherically symmetric solutions with a cosmological constant:** As we will see when we study cosmology (see also Carroll's Section 4.5), there is now pretty good evidence, from observations that the cosmic expansion is accelerating, for a cosmological constant Λ. What this means is that the Einstein equation gets modified to $G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$ (note that the sign of the *Lambda* term is determined by our choice of signature for the metric.) As we will see later, the cosmological constant Λ is sufficiently small so that it probably has no effects of sufficient magnitude to be observable in solar-system tests or pulsar-timing measurements, but it can be a big deal on cosmological scales.
   a. Solve for the most general spherically symmetric metric, in coordinates (t, r) that reduce to the ordinary Schwarzchild coordinates when $\Lambda = 0$.
   b. Write down the equation of motion for radial geodesics in terms of an effective potential, as in equation (5.66) in Carroll's book. Sketch the effective potential for massive particles.