General Relativity (Ph236a)
Problem Set 9
Due: December 5, 2006

Preview: There are a lot of substantial problems here, but you've also got a bit more time than usual. Problem 1 fills in mathematical details behind the Schwarzchild solution. You may want to make this lower priority if your time is limited. Problems 2, 3, and 4 should be fun; you get to work out what things look like if you're outside or falling into a Schwarchild black hole. Problem 5 has you work out the metric for a charged black hole. Problem 6 has you consider stars and spacetimes with cylindrical symmetry—this includes the spacetime around a cosmic string.

1. (Carroll, problem 4.5) **Hypersurface orthogonal timelike Killing vector:** A spacetime is static if there is a timelike Killing vector that is orthogonal to spacelike hypersurfaces.
   a. Generally speaking, if a vector field v^μ is orthogonal to a set of hypersurfaces defined by f =constant, then we can write the vector $v_\mu = h\nabla_\mu f$ (where both f and h are functions). Show that this implies that $v_{[\sigma}\nabla_\mu v_{\nu]} = 0$.
   b. Imagine we have a perfect fluid with zero pressure (dust), which generates a solution to Einstein's equations. Show that the metric can be static only if the fluid four-velocity is parallel to the timelike (and hypersurface-orthogonal) Killing vector.

2. (Carroll, problem 5.3) **Survival time for Schwarzchild black hole:** Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM$, of a Schwarzchild black hole. Use the ordinary Schwarzchild coordinates (t, r, θ, ϕ). Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{dt}\right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

   Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses. Show that this maximum proper time is achieved by falling freely with $E \to 0$.

3. (Carroll, problem 5.5) **Light signals from black holes:** Consider a comoving observer sitting at constant spatial coordinates (r_*, θ_*, ϕ_*) around a Schwarzchild black hole of mass M. The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength $\lambda_{\rm em}$ (in the beacon rest frame).
   a. Calculate the coordinate speed dr/dt of the beacon, as a function of r.
   b. Calculate the proper speed of the beacon. That is, imagine there is a comoving observer at fixed r, with a locally inertial coordinate system set up as the beacon passes by, and calculate the speed as measured by the comoving observer. What is it at $r = 2GM$?

c. Calculate the wavelength $\lambda_{\rm obs}$, measured by the observer at r_*, as a function of the radius $r_{\rm em}$ at which the radiation was emitted.
d. Calculate the time $t_{\rm obs}$ at which a beacm emitted by the beacon at radius $r_{\rm em}$ will be observed at r_*.
e. Show that at late times, the redshift grows exponentially with observed time: $\lambda_{\rm obs}/\lambda_{\rm em} \propto e^{t_{\rm obs}/T}$. Give an expression for the time constant T in terms of the black-hole mass M.

4. (From Lee Lindblom) **Tidal forces near black holes:** Use the geodesic deviation equation to derive an expression for the tidal force per unit length felt by a thin rigid object of length L and mass μ that falls radially into a Schwarzchild black hole of mass M. Assume that $L \ll M$ and that $\mu \ll M$. The breaking strength of standard 11-mm nylon climbing rope is about 2.5×10^6 g. How close could a 5000 cm length of such rope (with $\mu = 4000$ g) get to a 1 $M_\odot$ black hole before it breaks? For simplicity, assume that the rope does not stretch before it breaks.

5. **Charged black holes:** Consider the source-free ($j^\mu = 0$) Maxwell's equations in a static spherically symmetric spacetime.
   a. Argue that the most general form of an electromagnetic field-strength tensor that shares the static and spherical symmetries of the spacetime has nonzero components (in the usual Schwarzchild coordinates) $F_{tr} = -F_{rt} = f(r)$ and $F_{\theta\phi} = -F_{\phi\theta} = g(r)\sin\theta$, where $f(r)$ and $g(r)$ are functions of the radial coordinate r. By considering further the behavior at $r \to \infty$ and the fact (or assumption) that there are no magnetic monopoles, argue that $g(r) = 0$.
   b. Show that the solution of Maxwell's equations is $f(r) = -q/r^2$, where q is the charge of the black hole.
   c. Construct the stress-energy tensor $T_{\mu\nu}$ for this electromagnetic field configuration in the spacetime,

$$ds^2 = -A(r)\, dt^2 + B^2(r) dr^2 + r^2[d\theta^2 + \sin^2\theta\, d\phi^2].$$

   c. Follow the same steps used in the derivation of the Schwarzchild solution to show that the solution to the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ is the *Reissner-Nordstrom* spacetime,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2[d\theta^2 + \sin^2\theta\, d\phi^2].$$

   (Hint: You should find that the stress-energy tensor is traceless so that the Einstein equation reduces to $R_{\mu\nu} = 8\pi G T_{\mu\nu}$.)

6. **(2+1)-dimensional spacetimes:** Consider a static circularly symmetric (2+1)-dimensional spacetime, or equivalently, a cylindrical configuration in (3+1)-dimensions with perfect rotational symmetry.
   a. Show that there are vacuum solutions to Einstein's equations that can be written as

$$ds^2 = -dt^2 + \frac{1}{1 - 8GM} dr^2 + r^2 d\theta^2,$$

where M is a constant. (Incidentally, this is the metric around a cosmic string.)

b. Show that another way to write the same solution is

$$ds^2 = -d\tau^2 + d\xi^2 + \xi^2\, d\phi^2,$$

where $\phi \in [0, 2\pi(1 - 8GM)^{1/2}]$. (What you have thus shown is that the spacetime around a cosmic string is conical. It is flat—i.e., has zero curvature—but is missing a wedge.)

c. Now consider the spacetime around an infinite (in the $\mathbf{z}$ direction) cylinder of pressureless fluid and finite radius. Use the Newtonian limit to argue that the vacuum spacetime you computed above does not describe the spacetime around the cylinder. If you'd like, you can also try to find an exact solution to Einstein's equation, but I haven't tried this and don't know how hard it'll be. In this problem, you are showing, in other words, that the cylindrically symmetric vacuum solutions to Einstein's equations are not unique; i.e., that there is no Birkhoff's theorem for 2+1-d spacetimes. Can you explain why the cosmic string produces no curvature outside the source while the pressureless cylinder does produce curvature outside the source?