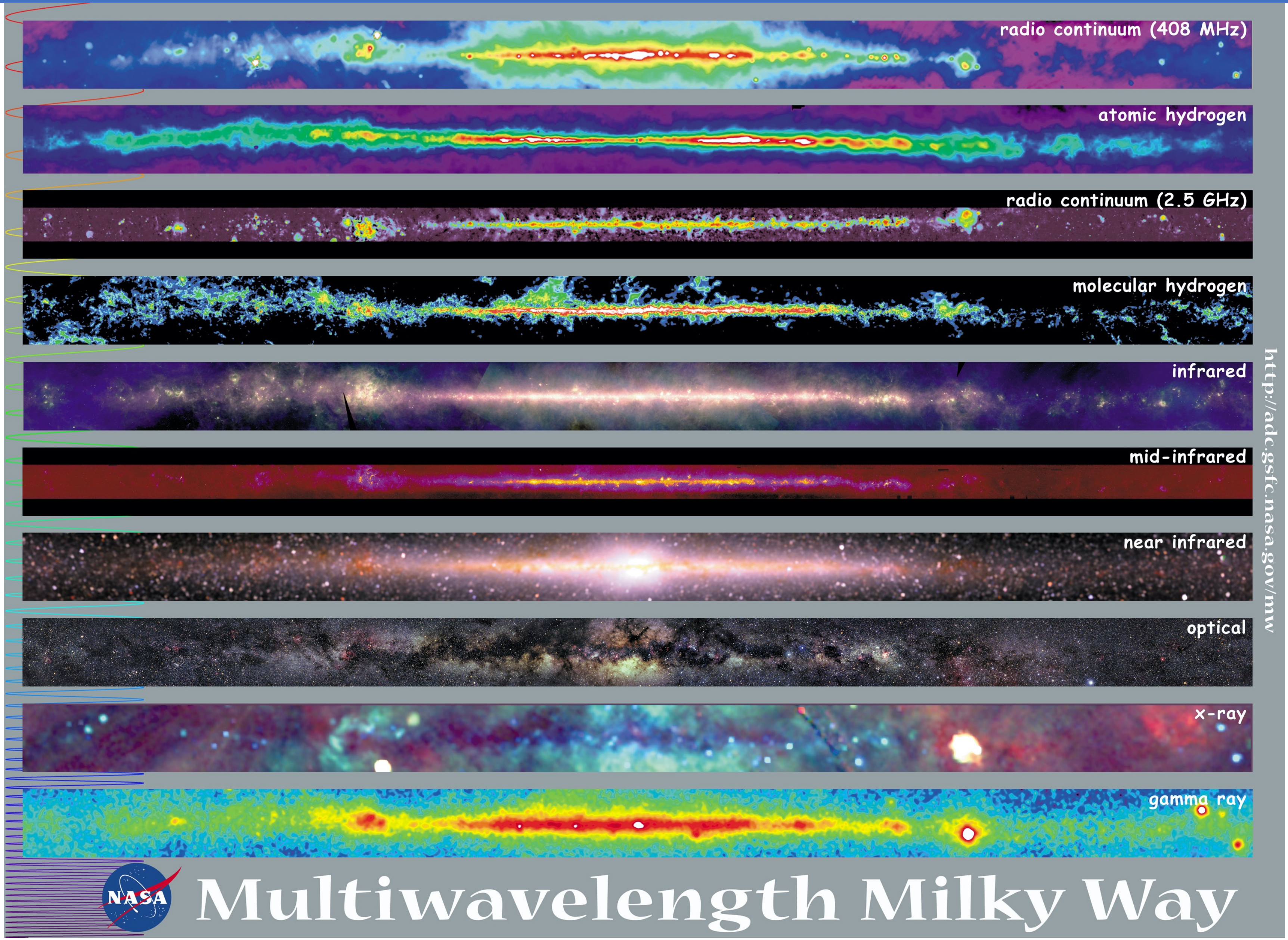
Welcome to Radiative Astrophysics (171.613)!



Radiative Astrophysics

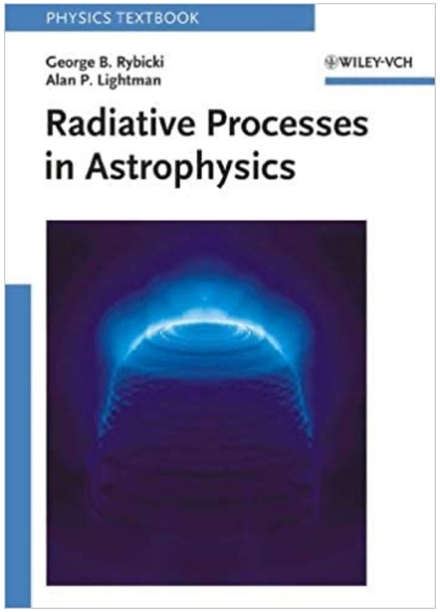
Instructor: David Neufeld (he/him/his – feel free to call me David)

Email: neufeld@jhu.edu

Class meetings: MW 1:30 – 2:45 pm except 9/5 (Labor Day)
Lecture notes will be posted on Canvas

Textbook: *Radiative Processes in Astrophysics* by Rybicki and Lightman

Available at the University bookstore or online (free, I think)
https://onlinelibrary.wiley.com/doi/book/10.1002/9783527618170



Course Requirements:

Homework: Problem sets will be handed out every week or two
Final exam: take home

Academic integrity:

The strength of the university depends on academic and personal integrity. In this course, you must be honest and truthful. Ethical violations include cheating on exams, plagiarism, reuse of assignments, improper use of the Internet and electronic devices, unauthorized collaboration, alteration of graded assignments, forgery and falsification, lying, facilitating academic dishonesty, and unfair competition.

IF YOU EVER HAVE ANY QUESTION ABOUT ANYTHING RELATED TO ACADEMIC INTEGRITY, ASK ME

Learning goals

Almost everything we know about the astrophysical Universe comes from observing electromagnetic radiation.

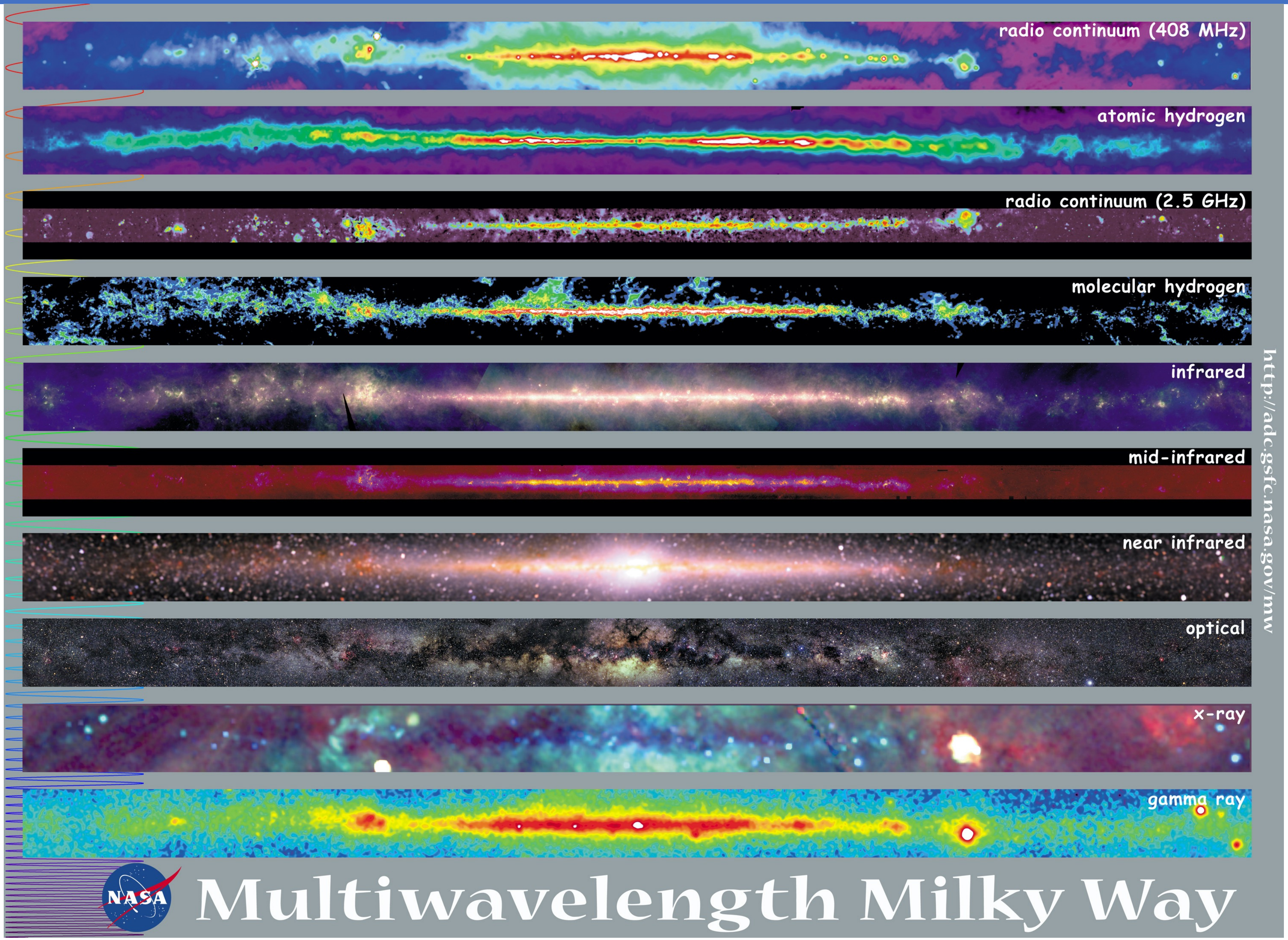
(Exceptions: some spacecraft investigations of the solar system; gravitational wave astrophysics)

The goals of this course are to

1) Learn about those physical processes that involve emission, absorption, and scattering of electromagnetic radiation

2) Appreciate how radiative processes *both* affect the nature of astrophysical objects *and* provide us with information about the astrophysical Universe

Multiwavelength views of the Galactic plane



Overall course structure

Part 1: Macroscopic description of radiation and of its propagation

(R&L, Chapter 1)

Part 2: Review and Extension of EM - the interaction of radiation with a point charge

(R&L, Chapters 2 - 4)

Part 3: Radiative Processes in Astrophysical Gas: Ionized Media

Bremsstrahlung, synchrotron radiation, Compton scattering, plasma effects

(R&L, Chapters 5 - 8)

Part 4: Radiative Processes in Astrophysical Gas: Atomic and Molecular Media

(R&L, Chapters 9 –11)

Radiative Astrophysics: schedule

1	Mon Aug 29	Introduction, Specific Intensity & Moments	R&L 1.1 – 1.3
2	Wed Aug 31	Radiative Transfer Equation & Moments	R&L 1.4
	Mon Sep 5	Labor Day: NO CLASS	
3	Wed Sep 7	Blackbody and Thermal Radiation	R&L 1.5
4	Mon Sep 12	Einstein Coefficients	R&L 1.6
5	Wed Sep 14	Scattering	R&L 1.7
6	Mon Sep 19	Radiative diffusion	R&L 1.8
7	Wed Sep 21	Maxwell's Eqns., Fourier Transforms	R&L 2.1 – 2.3
8	Mon Sep 26	Polarization	R&L 2.4
9	Wed Sep 28	EM Potentials and the L-W Potentials	R&L 2.5, 3.1
10	Mon Oct 3	Radiation fields, Dipole approx.	R&L 3.2
11	Wed Oct 5	Thomson Scattering, Harmonic Oscillator	R&L 3.4, 3.6
12	Mon Oct 10	Lorentz Transformations & 4-vectors	R&L 4.1, 4.2
13	Wed Oct 12	Emission from Relativistic Particles	R&L 4.8

Radiative Astrophysics: schedule

14	Mon Oct 17	Bremsstrahlung I	R&L 5
15	Wed Oct 19	Bremsstrahlung II	R&L 5
16	Mon Oct 24	Bremsstrahlung III	R&L 5
17	Wed Oct 26	Synchrotron Radiation I	R&L 6
18	Mon Oct 31	Synchrotron Radiation II	R&L 6
19	Wed Nov 2	Synchrotron Radiation III / Compton Scattering I	R&L 6
20	Mon Nov 7	Compton Scattering I	R&L 7
21	Wed Nov 9	Compton Scattering II	R&L 7
22	Mon Nov 14	Plasma Effects	R&L 8
23	Wed Nov 16	Atoms	R&L 9
	Mon Nov 21	Thanksgiving break: NO CLASS	
	Wed Nov 23	Thanksgiving break: NO CLASS	
24	Mon Nov 28	Atoms	R&L 9
25	Wed Nov 30	Radiative transitions	R&L 10
26	Mon Dec 5	Molecules	R&L 11

Lecture 1
Macroscopic description of radiation

Goal: understand the definitions of, and differences between

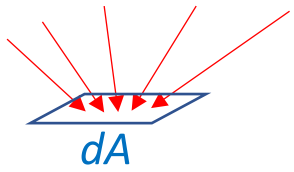
Radiative flux

Specific intensity

READING: *R&L* 1.1 – 1.3

Radiative flux

For an element of area, *dA*, the radiative flux (power per unit area) is defined by

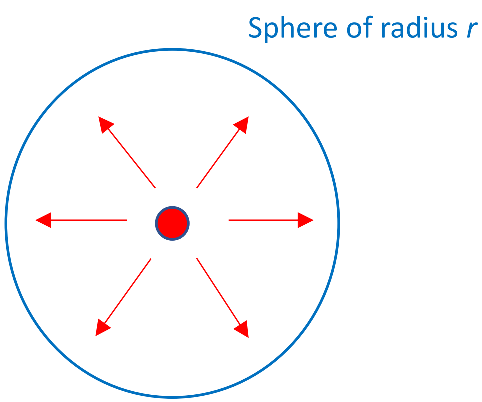


$$F = \frac{dE}{dAdt}$$

and has units: erg cm^{-2} s^{-1} (c.g.s.) = 10^{-3} W m^{-2} (SI)

Note that the flux passing through a given element depends on its orientation

For an isotropic source of luminosity (power) $L = dE/dt$, conservation of energy implies



$$F = \frac{L}{4\pi r^2}$$

Monochromatic flux

The flux carried by radiation in the frequency range ν to $\nu + d\nu$ can be written $F_\nu d\nu$

where

$$F_\nu = \frac{dE}{dA dt d\nu}$$

is the monochromatic flux

The c.g.s unit is erg cm^{-2} s^{-1} Hz^{-1}, and a commonly-used unit in astronomy is the Jansky (Jy)

1 Jy = 10^{-23} erg cm^{-2} s^{-1} Hz^{-1} = 10^{-26} W m^{-2} Hz^{-1}

Of course, we can also use wavelength in place of frequency and compute

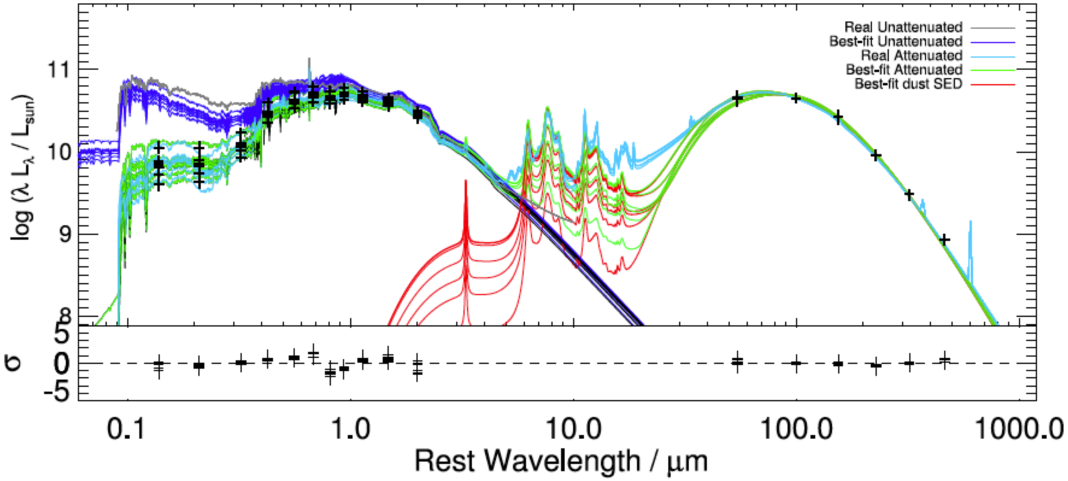
$$F_\lambda = \frac{dE}{dA dt d\lambda}$$

where $F_\lambda = F_\nu \, |d\upsilon / d\lambda|$ = $\upsilon^2 F_\nu$ / c or $\lambda F_\lambda = \nu F_\nu$

νF_ν

If you plot νF_ν versus lnν or lnλ, the area under the curve is the total flux

Example: Galaxy SED models from Hayward and Smith (2015)

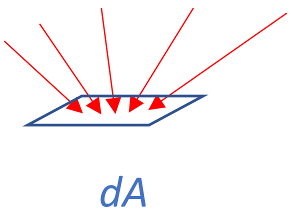


Two peaks at 1 and 100 micron indicate that stars and dust radiate roughly equal amounts of energy in this model

For a blackbody, $F \;=\; 1.36\,[\nu F_\nu]_{peak}$

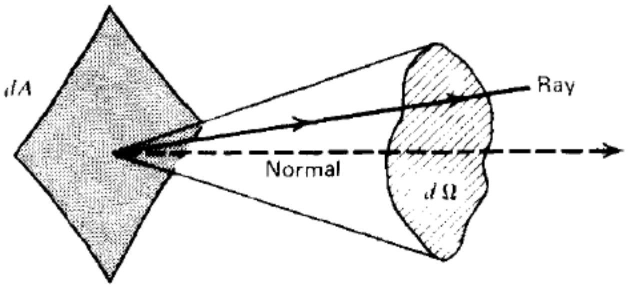
Specific intensity

Flux measures the total amount of radiation in all directions passing through an element of area



We can also think about a single ray (in the normal direction) and define the specific intensity

$$I_\nu = \frac{dE}{dA dt d\nu d\Omega}$$



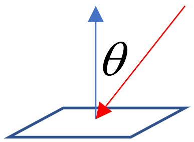
R&L Figure 1.2 Geometry for normally incident rays.

with units erg cm^{-2} s^{-1} Hz $^{-1}$ sr^{-1}

This defines the amount of energy within a cone of (infinitesimal) solid angle $d\Omega$

Relationship between flux and specific intensity

The flux can be considered an *angular moment* of the specific intensity, obtained from an integral over all directions



$$F_\nu dA = \int I_\nu dA' d\Omega$$

where dA' = cos$\theta\, dA$ is the *projected* area of the element of area

$$F_\nu = \int I_\nu \cos\theta\, d\Omega$$

Convenient notation: $\mu = \cos\theta$ ➔ $d\Omega = |\sin\theta\, d\theta\, d\phi| = |d\mu\, d\phi|$

$$F_\nu = \int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu \mu\, d\mu$$

Question 1 (in pairs; respond via Zoom poll): what is the flux for an isotropic radiation field (I_ν the same in all directions)?

Relationship between flux and specific intensity

If I_ν is independent of μ and ϕ, then

$$F_\nu = \int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu \mu \, d\mu$$

$$= 2\pi I_\nu \left[\mu^2/2\right]_{-1}^{1} = 0$$

Of course: the energy flow from top to bottom is exactly balanced by the flow from bottom to top

If uniformly bright radiation is incident over just one hemisphere ($\mu > 0$)

then $F_\nu = 2\pi I_\nu \left[\mu^2/2\right]_0^1 = \pi I_\nu$

Example: small window in a hot kiln

This is a useful result that we'll return to later

Lecture 2
Angular moments of specific intensity, radiative transfer

Goals: understand the significance of

Momentum flux / pressure

Mean intensity and energy density

Angular moments

Constancy of specific intensity along a ray (in vacuo), and the inverse square law for flux

READING: *R&L* 1.4, 1.5

Momentum flux

So far, we have been considering the flow of energy. But photons also carry momentum of magnitude *E/c*

Momentum is, of course, is a vector, and the normal component of momentum is

$E/c \cos\theta = E\mu / c$

The momentum flux is therefore

$$\int (I_\nu \mu / c) \cos\theta \, d\Omega = \int_0^{2\pi} d\phi \int_{-1}^{1} (I_\nu / c)\, \mu^2 \, d\mu$$

This quantity is proportional to the *second* angular moment of the intensity (whereas the flux is proportional to the *first* angular moment). This is the *pressure* (associated with radiation at frequency ν)

Question 1: for isotropic radiation, with intensity , I_ν, what is the pressure?

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Question 1: for isotropic radiation, with intensity , I_ν, what is the pressure?
Answer: $p_\nu = \ 2\pi I_\nu \, [\mu^3/3]_{-1}^{1} \, /c = 4\pi I_\nu / 3c$

Mean intensity

The energy flux and pressure are proportional to the first and second angular moments. What about the zeroth angular moment? This is just the mean (angle-averaged) intensity

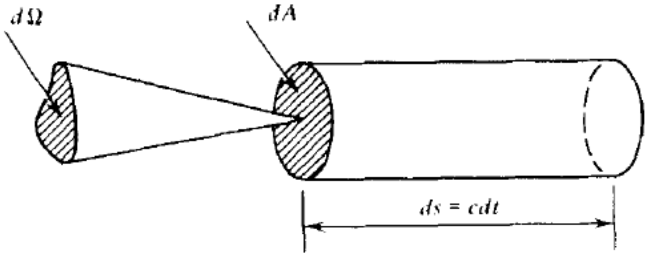
$$J_\nu = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu \, d\mu$$

…note the $1/(4\pi)$

This has the same units as specific intensity, and is proportional to the energy density, u_ν, of radiation at frequency ν

Element of energy in cylinder at right

$$\begin{aligned} dE &= I_\nu dA dt d\nu d\Omega \\ &= I_\nu dA (ds/c) d\nu d\Omega \\ &= I_\nu dV d\nu d\Omega / c \end{aligned}$$



R&L Figure 1.4 **Electromagnetic energy in a cylinder.**

So, the element of energy density is given by $du_\nu = dE/dV = I_\nu d\nu d\Omega / c$
Integrating over solid angle, we find

$$u_\nu = dE/dV = 4\pi J_\nu / c$$

Angular moments of I_ν

To make things more elegant, the $1/(4\pi)$ is typically used for all angular moments with the definitions

$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega$ "*Mean intensity*" $cu_\nu / 4\pi$

$H_\nu = \frac{1}{4\pi} \int I_\nu \cos\theta \, d\Omega$ "*Eddington flux*" $F_\nu / 4\pi$

$K_\nu = \frac{1}{4\pi} \int I_\nu \cos^2\theta \, d\Omega$ "*Second moment*" $cp_\nu / 4\pi$

For isotropic radiation,

$$H_\nu = 0$$

$K_\nu = \frac{1}{3} J_\nu$ ➔ pressure = $\frac{1}{3}$ energy density

(relativistic ideal gas)

Tensor representation

R&L treat the angular moments as scalars that are defined for an element of area in a specific orientation.

A more sophisticated analysis treats radiative flux ($4\pi H_\nu$) as a vector and pressure ($4\pi K_\nu/c$) as a 2nd-rank tensor

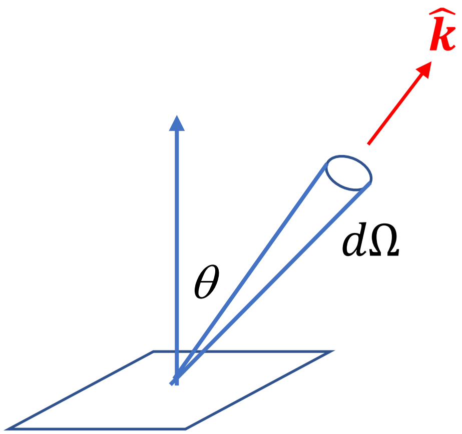
We define $\boldsymbol{H_\nu} = \frac{1}{4\pi}\int I_\nu \,\hat{\boldsymbol{k}}\, d\Omega$ and $\boldsymbol{K_\nu} = \frac{1}{4\pi}\int I_\nu \,\hat{\boldsymbol{k}}\hat{\boldsymbol{k}} d\Omega$

where $\hat{\boldsymbol{k}}$ is a unit directional vector.

In Cartesian coordinates $\hat{\boldsymbol{k}} = (\hat{k}_x, \widehat{k_y}, \hat{k}_z)$
$= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

So $\boldsymbol{H_\nu}$ has 3 components, $H_i = \frac{1}{4\pi}\int I_\nu \,\hat{k}_i d\Omega$,
where *i* = 1, 2, 3 for *x, y,* and *z*

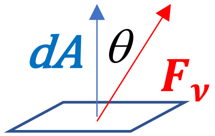
and $\boldsymbol{K_\nu}$ has 9 components, $K_{ij} = \frac{1}{4\pi}\int I_\nu \,\hat{k}_i\,\hat{k}_j d\Omega$



Tensor representation

With these definitions, the radiative flux, $\boldsymbol{F_\nu} = 4\pi \boldsymbol{H_\nu}$, is a vector that shows the *direction* in which the radiation is travelling

If we define a vector element of area $\boldsymbol{dA}$, where $\boldsymbol{dA}$ is pointing in the normal direction, energy passes through that area at a rate $\boldsymbol{F_\nu . dA}$



$\boldsymbol{F_\nu . dA} = F_\nu \, dA \cos\theta$

We can also write this using the "summation convention" in which we sum over repeated indices: $\frac{dE}{dt d\nu} = F_i \, dA_i$

The pressure, $\boldsymbol{p_\nu} = 4\pi \boldsymbol{K_\nu}/c$, is a 2nd-rank tensor, and the rate at which *j*-momentum passes through the element is the *j*-component of $p_{ij} \, dA_i$

Information content

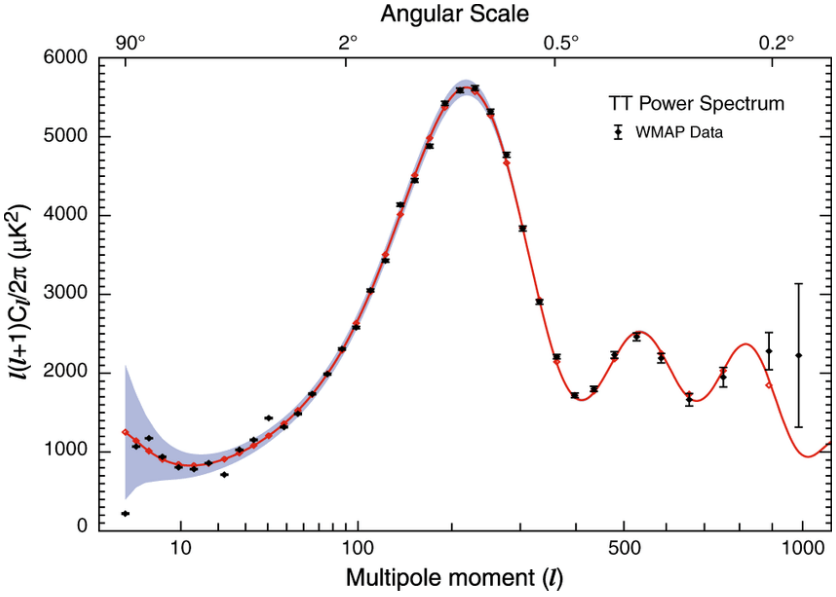
Successive angular moments provide increasing amounts of information about the angular distribution of the radiation. This is similar to a spherical harmonics expansion, in which we write

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi)$$

The coefficients a_{lm} provide successive finer detail about the angular distribution as l gets larger and larger

Aside: the spherical harmonics are related to the power spectrum (e.g. of the CMB), by the relation

$$C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2$$



Information content

In a spherical harmonics expansion:

There is one term with l = 0, corresponding to the one component of J_ν

There are 3 terms with l = 1, corresponding to the 3 components of $\boldsymbol{H}_\nu$

There are 5 terms with l = 2, but there are 9 components of $\boldsymbol{K}_\nu$

Question 2: but how many new pieces of information are provided by $\boldsymbol{K}_\nu$?

Information content

There are only *five* new pieces of information

Note first that

$K_{ij} = \frac{1}{4\pi} \int I_\nu \, \hat{k}_i \, \hat{k}_j d\Omega$ is symmetric, so there are only *six* independent components

Still, we haven't gotten to the *five* terms for l = 2

There is one additional relationship, involving the trace of $\boldsymbol{K}$

What is the trace of $\boldsymbol{K}$?

$$Tr(\boldsymbol{K}_{ij}) = K_{xx} + K_{yy} + K_{zz} = \frac{1}{4\pi} \int I_\nu \, (\hat{k}_x^2 + \hat{k}_y^2 + \hat{k}_z^2) d\Omega = J_\nu$$

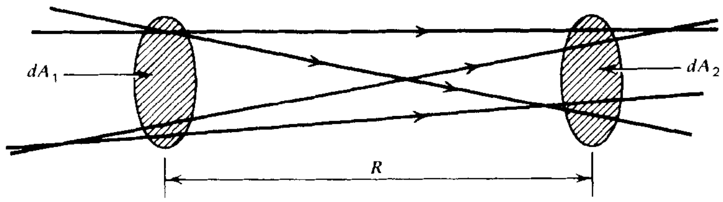
So this reduces the *new* information content by 1

General result: the first *q* angular moments provide the same directional information as the spherical harmonics expansion up to *l* = *q*

Radiative transfer: constancy of I_ν along a ray

In a vacuum, and in steady-state, the specific intensity is constant along a ray

Proof:



R&L *Figure 1.5 Constancy of intensity along rays.*

Consider the radiant energy that passes through both hoops shown above

$$dE = I_1 dA_1 dt d\nu d\Omega_2 = I_2 dA_2 dt d\nu d\Omega_1$$

where $d\Omega_{1,2} = dA_{1,2}/R^2$ is the solid angle subtended by hoop 1,2 as viewed from hoop 2,1

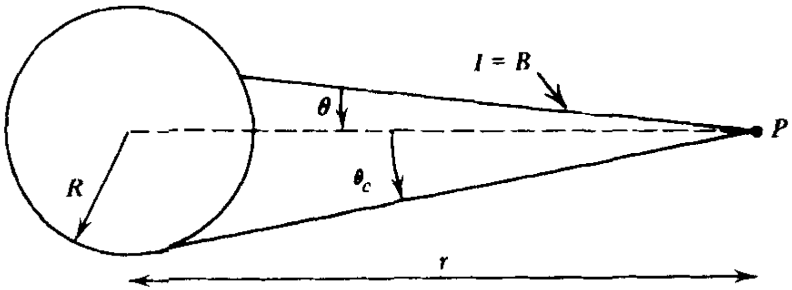
Hence $dA_1 d\Omega_2 = dA_2 d\Omega_1 \rightarrow I_1 = I_2$

If there's a time dependence, then $I_2(t + R/c) = I_1(t)$

How is this consistent with the inverse square law for flux?

Radiative transfer: constancy of I_ν along a ray

Consider now a uniformly bright ("Lambertian" sphere) that is viewed from a distance *r*



R&L ***Figure 1.6 Flux from a uniformly bright sphere.***

The observed flux is $F = \int_0^{2\pi} d\phi \int_{\cos\theta_c}^{1} \mu B \, d\mu$

$$= 2\pi B[\mu^2/2]_{\cos\theta_c}^{1} = \pi B(1 - \cos^2\theta_c) = \pi B \sin^2\theta_c$$

The intensity *B* does not depend on *r*, but the angle θ_c does!

$\sin\theta_c = R/r$ ➔ $F = \pi B\,(R/r)^2$

Constant intensity along a ray ➔ *inverse square law* for flux

Lecture 3
Radiative transfer

Goals: understand

Radiative transfer with emission and absorption

Optical depth and the source function

The radiative transfer equation as a "relaxation equation"

Angular moments of the transfer equation

Thermodynamics of blackbody radiation

READING: *R&L* 1.5

Emission

We now want to consider what happens when radiation travels through a medium capable of emitting and absorbing radiation

Define the spontaneous emission coefficient as the power emitted per unit volume per unit solid angle

$$j = \frac{dE}{dV dt d\Omega}$$

(units: erg cm^{-3} s^{-1} sr^{-1})

and the monochromatic emission coefficient in a similar way per unit bandwidth

$$j_\upsilon = \frac{dE}{dV dt d\Omega d\upsilon}$$

(units: erg cm^{-3} s^{-1} sr^{-1} Hz^{-1})

In travelling a distance *ds* along a ray, a beam of cross section *dA* travels through a volume $dV = dAds$, and the intensity therefore increases by $dI_\upsilon = j_\upsilon ds$

Other related quantity often used in stellar astrophysics:

$\epsilon_\upsilon =$ total monochromatic power per unit *mass* (erg s^{-1} g^{-1} Hz^{-1}) = $4\pi j_\upsilon/\rho$

Absorption

Absorption removes power from a ray in proportion to the intensity that is already present

Define absorption coefficient, α_υ, by the equation

$$\frac{dI_\upsilon}{ds} = -\alpha_\upsilon I_\upsilon$$

so α_υ has units cm^{-1}

Microscopic description: suppose we have *n* absorbing particles per unit volume, each of which presents a cross-section σ_υ to radiation at frequency ν

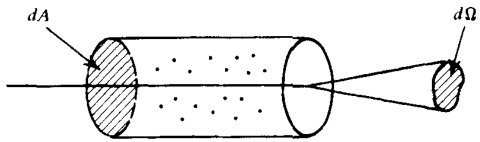
Number of particles in cylinder = *n dA ds*

Total cross-section presented = *n dA ds* σ_υ

Fraction of radiation removed = *n ds* σ_υ
(covering factor in lower panel at right)

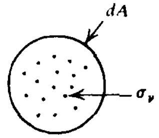
This fraction must equal $-dI_\upsilon / \, I_\upsilon$ = $\alpha_\upsilon ds$

which implies $\alpha_\upsilon = n\,\sigma_\upsilon$
(check dimensions: cm^{-1} = cm^{-3} cm^2 as required)



(a)

***Figure 1.7a** Ray passing through a medium of absorbers.*



(b)

***Figure 1.7b** Cross sectional view of 7a.*

Sign conventions and stimulated emission

The absorption coefficient, α_{υ}, is defined to be positive if the medium removes radiation in proportion to the amount already present

As we'll see later, there is an opposite process known as stimulated emission which *adds* radiation in proportion to the amount already present. This makes a negative contribution to the absorption coefficient.

Under ordinary circumstances, absorption beats stimulated emission and the combined effect yields a positive value of α_{υ}, leading to an exponential decay in the intensity:

$$\frac{dI_{\upsilon}}{ds} = -\alpha_{\upsilon}\, I_{\upsilon} < 0$$

But under special circumstances sometimes achieved in astrophysical media, stimulated emission can dominate absorption.

Then α_{υ} is negative and the intensity can *increase* exponentially (e.g. in maser = "*microwave amplification by the stimulated emission of radiation*")

$$\frac{dI_{\upsilon}}{ds} = -\alpha_{\upsilon}\, I_{\upsilon} > 0$$

Radiative transfer equation and optical depth

With both processes present, our equation becomes

$$\frac{dI_\upsilon}{ds} = j_\upsilon - \alpha_\upsilon\, I_\upsilon$$

Pure emission ($\alpha_\upsilon = 0$) solution: $I_\upsilon(s) = I_\upsilon(0) + \int_0^s j_\upsilon(s')ds'$

Pure absorption ($j_\upsilon = 0$) solution: $I_\upsilon(s) = I_\upsilon(0)\ \exp\left[-\int_0^s \alpha_\upsilon(s')ds'\right]$

With both processes present, it is convenient to define the "*optical depth*," τ_υ = $\int_0^s \alpha_\upsilon(s')ds'$. This is a dimensionless quantity.

The optical depth measures distance along the ray in units of the mean-free-path (i.e. the mean distance travelled before a photon gets absorbed)

$$l_{\text{mfp}} = 1/\alpha_\upsilon$$

Radiative transfer equation and source function

Noting that $\alpha_\upsilon ds = d\tau_\upsilon$, we may divide the transfer equation by α_υ to obtain

$$\frac{dI_\upsilon}{d\tau_\upsilon} = \frac{j_\upsilon}{\alpha_\upsilon} - I_\upsilon$$

We introduce the source function $S_\upsilon = j_\upsilon/\alpha_\upsilon$ to obtain

$$\frac{dI_\upsilon}{d\tau_\upsilon} = S_\upsilon - I_\upsilon$$

This is a relaxation equation, in that the intensity is relaxing towards S_υ as it moves along the ray (although it may never reach S_υ)

In other words, if $I_\upsilon < S_\upsilon$ it will increase whereas if $I_\upsilon > S_\upsilon$ it will decrease

The formal solution is $$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu)\,d\tau'_\nu.$$

In an optically-thick medium with $\tau_\upsilon >> 1$, the first term is very small and there is no "memory" of $I_\upsilon(0)$. I_υ gets very close to the local source function

Angular moments of the transfer equation

So far we've written the radiative transfer equation for a single ray

$$\frac{dI_v}{ds} = j_v - \alpha_v\, I_v$$

where s is the distance along that ray

We can easily write this for all rays at once

$$\hat{\boldsymbol{k}}\,.\boldsymbol{\nabla} I_v(\hat{\boldsymbol{k}}) = \; j_v - \alpha_v\, I_v(\hat{\boldsymbol{k}})$$

And add in time dependence if needed (rarely)

$$\frac{1}{c}\frac{\partial I_v}{\partial t} + \hat{\boldsymbol{k}}\,.\boldsymbol{\nabla} I_v(\hat{\boldsymbol{k}}) = \; j_v - \alpha_v\, I_v(\hat{\boldsymbol{k}})$$

Later: look back at Lecture 2 and convince yourself I did this right

Angular moments of the transfer equation

Let's take angular moments of this equation, i.e. multiply by μ^n and integrate $d\Omega$

$$\frac{1}{c}\frac{\partial I_v}{\partial t} + \widehat{\boldsymbol{k}}\,.\boldsymbol{\nabla} I_v(\widehat{\boldsymbol{k}}) = j_v - \alpha_v\, I_v(\widehat{\boldsymbol{k}})$$

Zeroth moment ($n = 0$):

$$\frac{4\pi}{c}\frac{\partial J_v}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{F_v} = 4\pi j_v - 4\pi\alpha_v\, J_v$$

where I've assumed that α_v is isotropic and noted

$$\int \widehat{\boldsymbol{k}}\,.\boldsymbol{\nabla} I_v(\widehat{\boldsymbol{k}})d\Omega = \int \hat{k}_i\,\frac{\partial I_v}{\partial \hat{k}_i}d\Omega = \frac{\partial}{\partial \hat{k}_i}\int \hat{k}_i\, I_v\, d\Omega = \frac{\partial F_{vi}}{\partial \hat{k}_i} = \boldsymbol{\nabla}.\boldsymbol{F_v}$$

Above, we are using the summation convention in which we sum over repeated indices, i.e. we abbreviate $\sum_i x_i\, x_i$ as $x_i\, x_i$

Question: what physical principle does this differential equation encapsulate? Explain your answer briefly.

Angular moments of the transfer equation

The zeroth moment of the transfer equation is a statement of energy conservation

Time derivative of the density of a conserved quantity

Divergence of the flux of that quantity

$$\frac{\partial u_v}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{F}_v = 4\pi j_v - 4\pi\alpha_v J_v$$

Rate of production per unit volume ("sources")

Rate of destruction per unit volume ("sinks")

Angular moments of the transfer equation

Let's take angular moments of this equation, i.e. multiply by μ^n and integrate $d\Omega$

$$\frac{1}{c}\frac{\partial I_\upsilon}{\partial t} + \hat{\boldsymbol{k}}\,.\boldsymbol{\nabla} I_\upsilon(\hat{\boldsymbol{k}}) = j_\upsilon - \alpha_\upsilon\, I_\upsilon(\hat{\boldsymbol{k}})$$

First moment (n = 1):

$$\frac{1}{c}\frac{\partial \boldsymbol{F_\upsilon}}{\partial t} + 4\pi\, \boldsymbol{\nabla}.\boldsymbol{K}_\upsilon = 0 - \alpha_\upsilon\ \boldsymbol{F_\upsilon}$$

where we note that

$$\int \hat{\boldsymbol{k}}\,.\boldsymbol{\nabla} I_\upsilon(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{k}} d\Omega\ = \int \hat{k}_i \frac{\partial I_\upsilon}{\partial \hat{k}_i}\hat{k}_j\, d\Omega\ = 4\pi \frac{\partial K_{\upsilon ij}}{\partial \hat{k}_i} = 4\pi\, \boldsymbol{\nabla}.\boldsymbol{K}_\upsilon$$

Divergence of a 2nd-rank tensor is a vector
Question: what physical principle does this differential equation encapsulate? Explain your answer briefly.

Angular moments of the transfer equation

Dividing through by *c*, we see that first moment of the transfer equation is a statement of momentum conservation

Time derivative of the momentum density

Divergence of the momentum flux

$$\frac{1}{c^2}\frac{\partial \boldsymbol{F_v}}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{p_v} = -\alpha_v \boldsymbol{F_v}/\boldsymbol{c}$$

Rate of “destruction” per unit volume = force per unit volume exerted by the radiation field on the gas

Blackbody radiation: thermodynamic considerations

Blackbody radiation is radiation in thermal equilibrium

Allow an isolated enclosure to reach TE, i.e. a state of maximum entropy

$$I_v = B_v(T)$$

Theorem: I_v is a universal function, of v and *T*, which is independent of direction and the nature of the enclosure (shape, material…). This we call the *Planck function*, $B_v(T)$

Blackbody radiation: thermodynamic considerations

Proof: Join the enclosure to another enclosure at the same temperature, with a filter that reflects all radiation except at frequency υ. Radiation at frequency υ can pass between the enclosures through a hole.

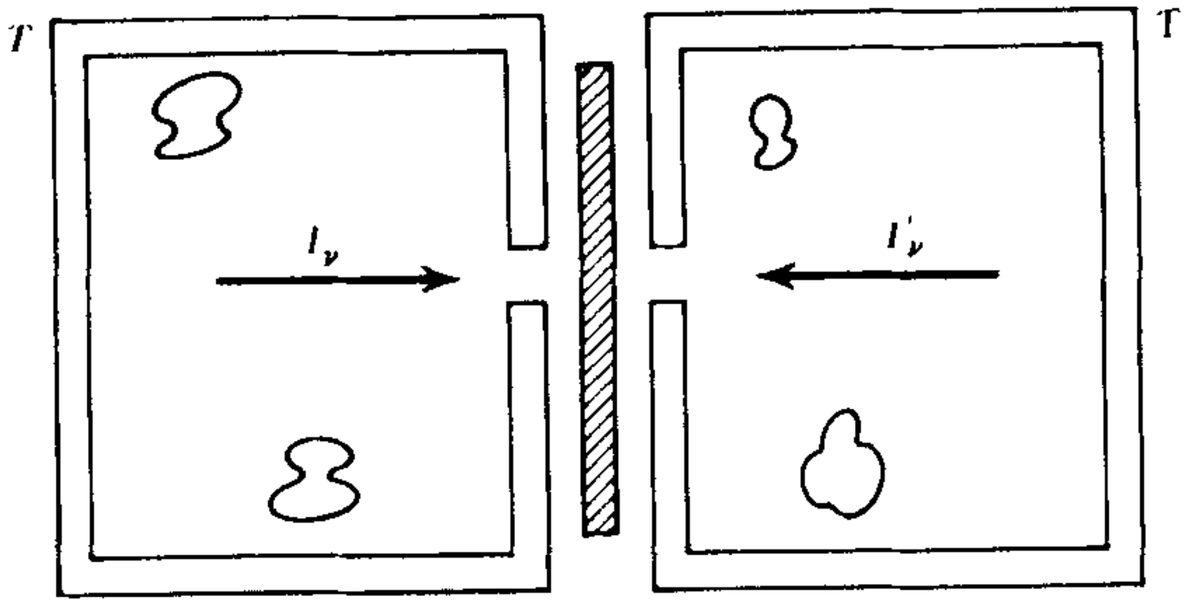
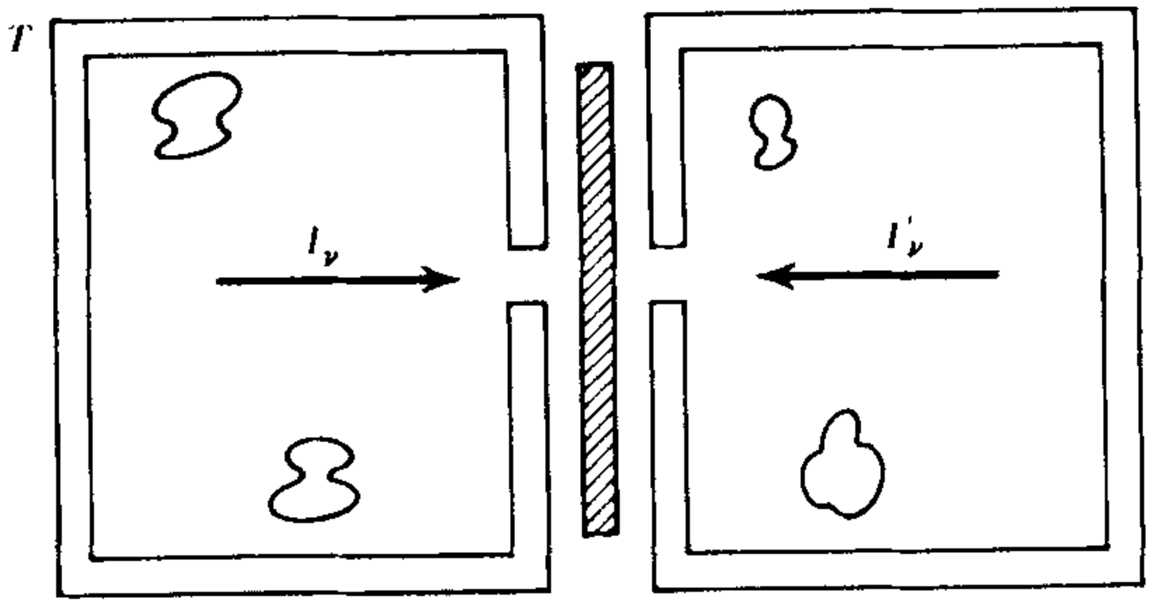


Figure 1.8 Two containers at temperature T, separated by a filter. ***(R&L)***

Unless $I_\upsilon = I_\upsilon{}'$ at all frequencies (see Figure) and angles, energy could pass from one enclosure to the other, violating the 2nd Law of Thermodynamics

Blackbody radiation: thermodynamic considerations

Corollary: The universal function, B_υ, must be a monotonically increasing function of *T* at every frequency



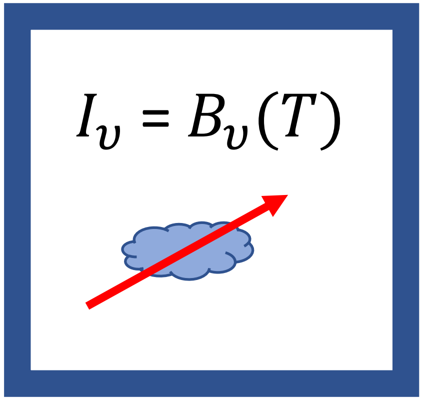
T' > T

Two containers at different temperatures

Unless $I'_\upsilon \geq I_\upsilon$ heat could pass from the cooler container (left) to the hotter, violating the 2nd Law of Thermodynamics. (Condition must be satisfied at every frequency)

Kirchhoff's Law for material in TE

Suppose we have a blob of material inside the enclosure (in equilibrium, so at temperature, *T*)



In thermal equilibrium, $I_\nu = B_\nu(T)$, so

$$0 = \frac{dI_\nu}{ds} = S_\nu - I_\nu = S_\nu - B_\nu$$

➔ $S_\nu = B_\nu(T)$ in TE, or $j_\nu = \alpha_\nu\, B_\nu(T)$

This is *Kirchhoff's Law*, which relates the absorption and emission coefficients in TE

Kirchhoff's Law

Kirchhoff's Law applies to any material, under TE conditions. The latter means that the material at temperature *T* is surrounded by blackbody radiation at the same temperature.

However, in many circumstances, the material properties are not affected by the radiation that it is exposed to. In that case, Kirchhoff's Law may still apply, and the material is said to be in *local* thermodynamic equilibrium (LTE)

Example: glass rod heated with a Bunsen burner (but not inside a furnace). To a good approximation, $j_v = \alpha_v \, B_v(T)$

Kirchhoff's law implies that good absorbers are good emitters, and poor absorbers are poor emitters. This is why stainless steel makes a good teapot material (low absorptivity in the thermal IR)

Lecture 4
Blackbody radiation, Einstein coefficients

Goals: understand

The statistical mechanics of blackbody radiation

The Planck function

The Einstein coefficients

The equations of statistical equilibrium

Alternative textbook:

The Physics of Astrophysics Volume 1: Radiation by Frank Shu

ISBN: 978-1891389764

Statistical mechanics and the Planck function

To derive the *Planck function*, $B_\upsilon(T)$, we start with four basic principles concerning photons

1) Each photon has energy $h\nu$

2) There's a finite density of quantum states in phase space, $dN_q / (d^3p\, d^3x) = 1/h^3$

When we account for the fact that photons have spin = 1 and two possible helicities, this becomes $2/h^3$

3) Since photons are bosons, there is no limit on the number of photons that can occupy a given quantum state. This number is called the photon occupation number, $\mathcal{N}$

4) In thermal equilibrium, the probability of finding *n* photons in any given state is proportional to the *Boltzmann factor* $\exp(-E_n/kT)$, where $E_n = nh\nu$

Statistical mechanics: photon occupation number, $\mathcal{N}$

The probability that a given quantum state contains *n* photons is therefore

$$\Pr(n) = \frac{e^{-nh\nu/kT}}{Z}$$

where the "partition function," *Z*, is the quantity needed to normalize the probabilities so they sum to unity

$$Z = \sum_{n=0}^{\infty} e^{-nh\nu/kT} = \frac{1}{1 - e^{-h\nu/kT}}$$

The mean occupation number is therefore

$$\mathcal{N} = \sum_{n=0}^{\infty} n \Pr(n) = \frac{1}{Z} \sum_{n=0}^{\infty} n e^{-nh\nu/kT} = \frac{1}{Z} \frac{dZ}{d\left(\frac{h\nu}{kT}\right)}$$

$$\mathcal{N} = \left(1 - e^{-h\nu/kT}\right) \frac{e^{-h\nu/kT}}{\left(1 - e^{-h\nu/kT}\right)^2} = \frac{1}{e^{h\nu/kT} - 1}$$

Energy density and Planck function

We're now ready to compute the energy density of radiation at frequency ν to $\nu+d\nu$

$$u_\nu d\nu = h\nu \left(\frac{2}{h^3}\right) \mathcal{N}\, d^3p = h\nu \left(\frac{2}{h^3}\right) \mathcal{N}\, 4\pi p^2 dp$$

Energy per photon Density of photons in phase space

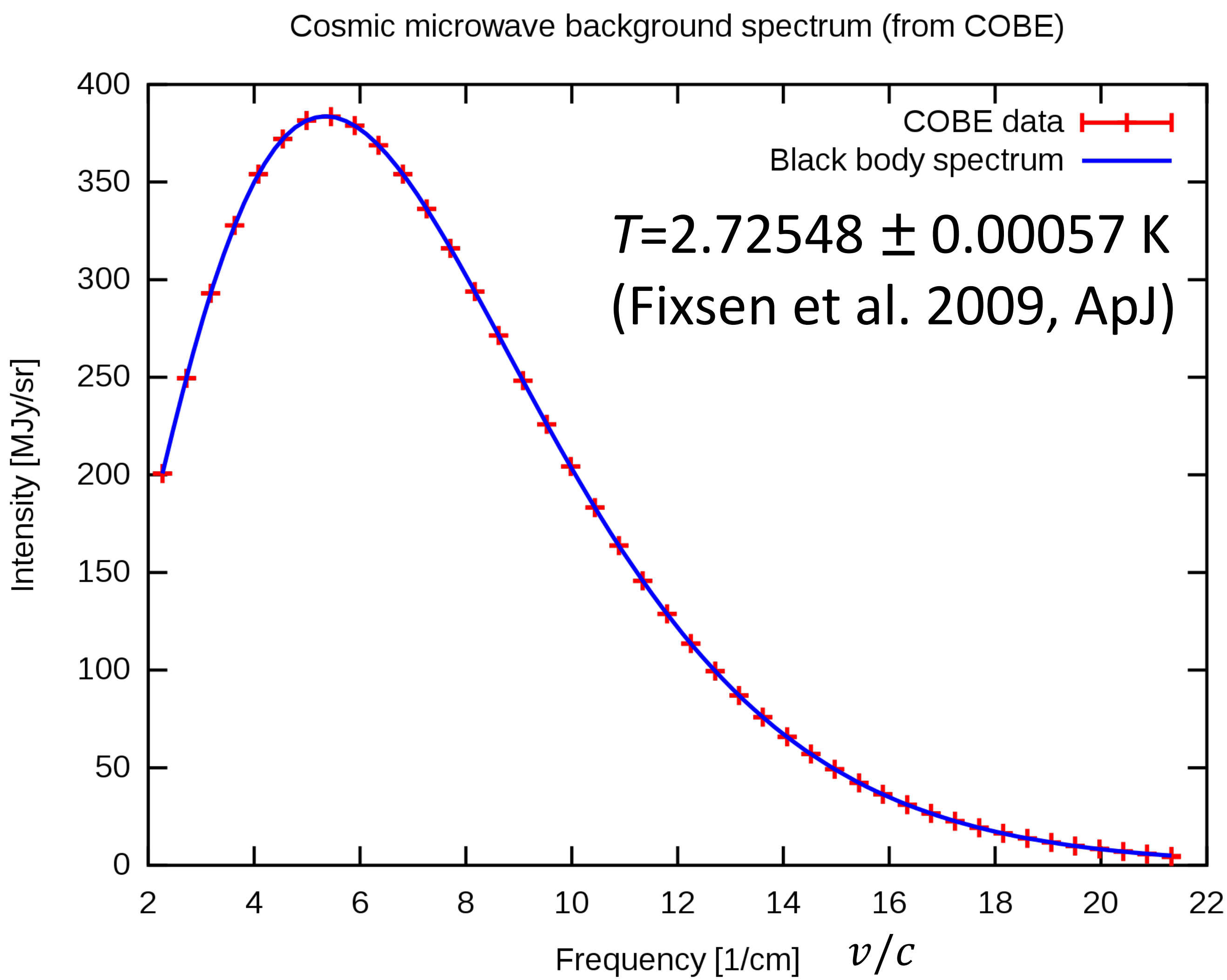
The momentum of a photon has magnitude p $= h\nu/c$, so this becomes $u_\nu d\nu = 8\pi h\nu \left(\frac{\nu^2}{c^3}\right) \mathcal{N}\, d\nu$

➔ $J_\nu = cu_\nu/4\pi = \left(\frac{2h\nu^3}{c^2}\right) \mathcal{N}$

➔ $B_\nu = \left(\frac{2h\nu^3}{c^2}\right) \frac{1}{e^{h\nu/kT}-1}$

The Cosmic Microwave Background

The CMB provides a beautiful example of a Planck function, and is the most accurately measured in any experiment



Significance of energy quantization

Note that we cannot derive the Planck function without discussing photons, i.e. the fact that radiant energy at frequency ν is quantized in multiples of $h\nu$

Without this (i.e. in the limit $h \to 0$), we obtain the "classical result," which diverges in the limit of large ν (the "*ultraviolet catastrophe*")

$$B_\nu = \left(\frac{2h\nu^3}{c^2}\right)\frac{1}{e^{h\nu/kT}-1} \to \left(\frac{2kT\nu^2}{c^2}\right) = \frac{2kT}{\lambda^2}$$

This limit does indeed apply in the limit of low frequency ($h\nu/kT \ll 1$) and is called the *Rayleigh-Jeans Law*

In the opposite limit, B_ν is well approximated by $\frac{2h\nu^3}{c^2}e^{-h\nu/kT}$ (*"Wien's Law")*

Properties of the Planck function: derivatives

$$B_v = \left(\frac{2hv^3}{c^2}\right)\frac{1}{e^{hv/kT}-1} \qquad \text{or} \qquad B_\lambda = \left(\frac{2hc^2}{\lambda^5}\right)\frac{1}{e^{hc/\lambda kT}-1}$$

Clearly $\partial B_v/\partial T > 0$ for all T and v, as required by the Second Law of Thermodynamics

Solving for $\partial B_v/\partial v = 0$ (yields a transcendental eqn.), we may determine where the Planck function peaks

$$\frac{v_{\rm peak}}{T} = \frac{2.81k}{h} = 58.8\ \mathrm{GHz/K}$$

Solving for $\partial B_\lambda/\partial\lambda = 0$, we get the Wien displacement law

$$\lambda_{\rm peak}T = \frac{4.97hc}{k} = 0.290\ \mathrm{cm\ K}$$

Wien displacement law

Source	Temperature	λ_{peak}	Waveband
CMB	2.73 K	1.1 mm	mm-wave
Pluto	44 K	66 μm	Far-IR
Earth	287 K	10 μm	Mid-IR
0.1 $M_\odot$ main sequence star	2900 K	1.0 μm	Near-IR
Sun	5800 K	500 nm (5000 Å)	Visible
10 $M_\odot$ main sequence star	20000 K	145 nm (1450 Å)	Far-UV
Youngest white dwarfs	250,000 K	12 nm (120 Å)	Extreme UV / soft X-ray

Properties of the Planck function: integrals

$$B_\nu = \left(\frac{2h\nu^3}{c^2}\right)\frac{1}{e^{h\nu/kT}-1}$$

Integrating over frequency with the change of variable $x = h\nu/kT$, we obtain

$$B = \int_0^\infty B_\nu \, d\nu = \left(\frac{2h}{c^2}\right)\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx$$

$$= \left(\frac{2h}{c^2}\right)\left(\frac{kT}{h}\right)^4 \left(\frac{\pi^4}{15}\right)$$

For blackbody radiation leaving a surface isotropically, the flux is $F = \pi B = \sigma_{\rm sb} T^4$ (the "*Stefan-Boltzmann Law*")

where $\sigma_{\rm sb} = \left(\frac{2h}{c^2}\right)\left(\frac{k}{h}\right)^4 \left(\frac{\pi^5}{15}\right) = 5.67 \times 10^{-5}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}$

Properties of the Planck function: integrals

The total energy density is

$$u = \frac{4\pi B}{c} = \frac{4\sigma_{\mathrm{sb}} T^4}{c} = aT^4$$

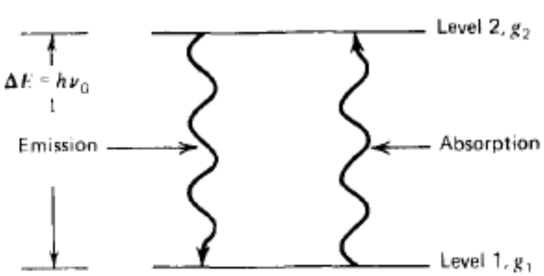
where $a = 7.57 \times 10^{-15}$ erg cm^{-3} K^{-4} is called the *"radiation constant"*

The pressure associated with blackbody radiation can dominate in the interiors of high-mass stars

$$p = \frac{aT^4}{3}$$

The Einstein coefficients

We consider an atom or molecule with two states, 1 and 2



R&L ***Figure 1.12a Emission and absorption from a two level atom.***

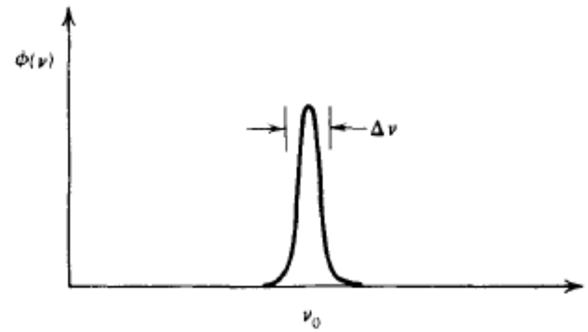


Figure 1.12b Line profile for 12a.

Spontaneous emission

A_{21} = transition probability per unit time for spontaneous emission (sec^{-1}).

Atoms per unit volume in state 2

$$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu)$$

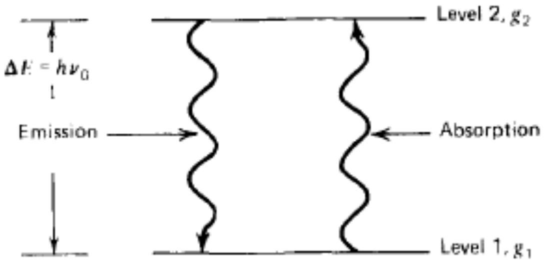
Line profile function (units Hz^{-1}), normalized such that

$$\int_0^\infty \phi(\nu)\, d\nu = 1.$$

$\phi(\nu)$ is not quite a delta function, because various processes (natural linewidth, Doppler motions) give the line a finite width, but typically $\Delta\nu \ll \nu_0$

The Einstein coefficients

We consider an atom or molecule with two states, 1 and 2



R&L Figure 1.12a Emission and absorption from a two level atom.

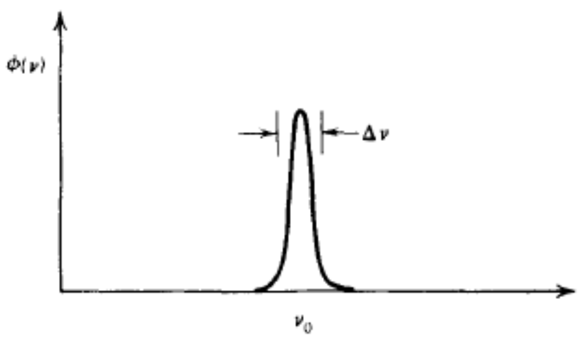


Figure 1.12b Line profile for 12a.

<u>Absorption</u>

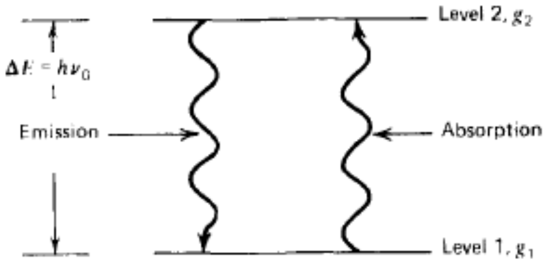
$B_{12}\bar{J}$ = transition probability per unit time for absorption

$$\bar{J} \equiv \int_0^\infty J_\nu \phi(\nu)\, d\nu$$

If J_ν varies slowly with ν and $\Delta\nu \ll \nu_0$, then we ***can*** treat $\phi(\nu)$ as a delta function and write $\bar{J} = J_\nu(\nu_0)$

The Einstein coefficients

We consider an atom or molecule with two states, 1 and 2



R&L ***Figure 1.12a Emission and absorption from a two level atom.***

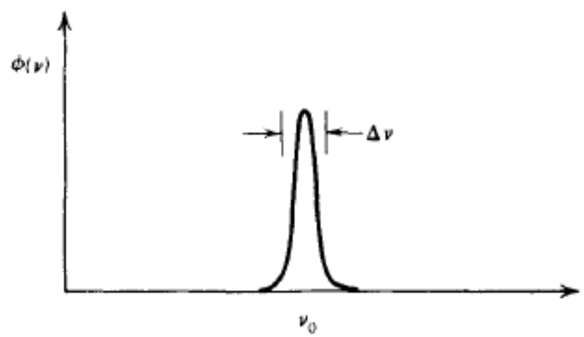


Figure 1.12b Line profile for 12a.

Stimulated emission

$B_{21}\bar{J}$ = transition probability per unit time for stimulated emission.

The process is the reverse of absorption

Unlike spontaneous emission, its rate is *proportional* to the radiation field

As we'll see (and as Einstein found), this process *has* to be present on thermodynamic grounds

Equation of statistical equilbrium

In steady-state, the rate of transitions from 2 to 1 will be exactly balanced by the rate from 1 to 2

$$n_2 A_{21} + n_2 B_{21} \bar{J} = n_1 B_{12} \bar{J}$$

which implies

$$\bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$

Now, suppose we are in thermal equilibrium

Then $\bar{J} = B_\upsilon$ (Planck function)

and $\frac{n_1}{n_2} = \frac{g_1 \exp(-E/kT)}{g_2 \exp[-(E + h\nu_0)/kT]} = \frac{g_1}{g_2} \exp(h\nu_0/kT)$ (Boltzmann factor)

Equation of statistical equilibrium

Given the Boltzmann factor for n_2/n_1, we then have

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/kT) - 1}$$

If $\bar{J}$ is to equal the Planck function, $\left(\frac{2h\nu^3}{c^2}\right)\frac{1}{e^{h\nu/kT}-1}$

we require $$g_1 B_{12} = g_2 B_{21},$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Emission and absorption coefficients

As noted previously, $$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu)$$

This is the rate at which energy is added by spontaneous emission (per unit bandwidth, per unit volume, per unit solid angle)

The combined effect of absorption and stimulated emission is to remove energy at a rate proportional to the mean intensity, $[n_1 B_{12} - n_2 B_{21}]\, h\nu\, \phi(\nu)\, J_\nu$ (per unit bandwidth, per unit volume)

This has to equal $\int \alpha_\nu I_\nu \, d\Omega = 4\pi \alpha_\nu J_\nu$

so $\alpha_\nu = \frac{h\nu}{4\pi} [n_1 B_{12} - n_2 B_{21}]\, \phi(\nu)$

The source function

Given these expressions for α_ν and j_ν , and using the relationship between the Einstein coefficients, we may compute the source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2\ A_{21}}{n_1\ B_{12}\ - n_2\ B_{21}} = \frac{2h\nu^3}{c^2}\frac{1}{\frac{n_1 g_2}{n_2 g_1}-1}$$

In thermal equilibrium, $\frac{n_1 g_2}{n_2 g_1}$ = $e^{h\nu/kT}$ (Boltzmann factor)

and we recover the Kirchhoff's Law, S_ν= B_ν

The condition for "local thermodynamic equilibrium" (LTE) is clearly just that $n_1/\ n_2$ is given by the Boltzmann factor

This typically holds at sufficiently high density, regardless of whether J_ν= B_ν

Example: the surface of the Sun, where J_ν~ $\frac{1}{2} B_\nu$ but $n_1/\ n_2$ is typically close to LTE

Lecture 5

Goals: understand

Different types of “temperature”

Effects of collisional excitation

The kinetic and excitation temperatures

Maser amplification and its astrophysical applications

The excitation temperature

Regardless of whether a system is in LTE, we can *always define some temperature,* such that

$$\frac{n_1 g_2}{n_2 g_1} \equiv e^{hv/kT_{\mathrm{ex}}}$$

We call this the “excitation temperature,” T_{ex}

The condition for LTE is then that T_{ex} equals the temperature of the gas

Either way, $S_v = B_v(T_{\mathrm{ex}})$

Other temperatures astronomers like to define

We can also define several other temperatures that are equal to each other in TE but could differ in other circumstances. These *definitions* are used regardless of whether we are in TE

Kinetic temperature, T_{kin}, characterizes the distribution of particle velocities (Maxwell-Boltzmann distribution at temperature T_{kin})

Radiation temperature, $T_{rad}(\nu)$ characterizes the mean intensity through the definition $J_\nu \equiv B_\nu(T_{rad})$

Brightness temperature, $T_B(\nu)$, characterizes the specific intensity along a given ray through the definition $I_\nu \equiv B_\nu(T_B)$

Rayleigh-Jeans brightness temperature, $T_{B,RJ}$, characterizes the specific intensity along a given ray through the definition $I_\nu \equiv 2kT_{B,RJ}/\lambda^2$. This is used *whether or not the RJ limit applies.*

Other temperatures

Effective temperature, T_{eff}, is a measure of total (i.e. frequency integrated) flux, through the definition $F \equiv \sigma_{SB} T_{eff}^4$)

This is widely used in describing stars, for which the luminosity may be written $L = 4\pi R^2 \sigma_{SB} T_{eff}^4$

In LTE, $T_{ex} = T_{kin}$ ➔ Kirchhoff's Law holds since

$$S_\nu = B_\nu(T_{ex}) = B_\nu(T_{kin})$$

In complete thermal equilibrium, $T_{ex} = T_{kin} = T_{rad}$

For isotropic radiation T_B (along any ray) $= T_{rad}$

But note that $T_{B,RJ}$ differs from T_B unless $h\nu << kT_B$

Effect of collisions on the excitation temperature

With radiative processes alone, we had

$$n_2\, A_{21} + \; n_2\, B_{21}\, \bar{J} = n_1\, B_{12}\, \bar{J}$$

But inelastic/superelastic collisions with another particle can also induce a transition from one state to another

These are characterized by a collision rate that depends on the *kinetic* temperature

C_{12} = rate of inelastic collisions from state 1 to 2
C_{21} = rate of superelastic collisions from state 2 to 1

We now have

$$n_2\, C_{21} + n_2\, A_{21} + \; n_2\, B_{21}\, \bar{J} = \; n_1\, C_{12} + \; n_1\, B_{12}\, \bar{J}$$

Effect of collisions on the excitation temperature

We now have

$$\frac{g_2}{g_1}\exp(-h\nu/kT_{\mathrm{ex}}) \equiv \frac{n_2}{n_1} = \frac{C_{12} + B_{12}\bar{J}}{C_{21} + A_{21} + B_{21}\bar{J}}$$

Let's consider first the case where collisions are absent (say because the density is very low) and we are in thermal equilibrium at temperature, *T*

In thermal equilibrium at temperature *T*, we know that radiative processes alone give us

$$\frac{g_2}{g_1}\exp(-h\nu/kT) = \frac{B_{12}\bar{J}}{A_{21} + B_{21}\bar{J}}$$

The right hand side depends only on the radiation field, so this must mean

$$\frac{B_{12}\bar{J}}{A_{21} + B_{21}\bar{J}} = \frac{g_2}{g_1}\exp(-h\nu/kT_{\mathrm{rad}})$$

whether or not we are in TE

Effect of collisions on the excitation temperature

Now suppose we are in TE and collisions are significant

If $\frac{g_2}{g_1}\exp(-h\nu/kT)$ is to equal $\frac{C_{12}+B_{12}\bar{J}}{C_{21}+A_{21}+B_{21}\bar{J}}$, then we must also have

$$\frac{C_{12}}{C_{21}} = \frac{g_2}{g_1}\exp(-h\nu/kT)$$

Since collisions are controlled by the kinetic temperature of the gas, this must mean $\frac{C_{12}}{C_{21}} = \frac{g_2}{g_1}\exp(-h\nu/kT_{\mathrm{kin}})$ *whether or not we are in TE*

In general, $T_{\mathrm{kin}} \neq T_{\mathrm{rad}}$, then

$$\frac{n_2}{n_1} = \frac{C_{12} + B_{12}\bar{J}}{C_{21} + A_{21} + B_{21}\bar{J}}$$

will lie somewhere between $\frac{g_2}{g_1}\exp(-h\nu/kT_{\mathrm{kin}})$ and $\frac{g_2}{g_1}\exp(-h\nu/kT_{\mathrm{rad}})$

"Critical density"

The collision rates, C_{21} and C_{12}, are proportional to the volume density of particles, n, with which our atom can collide, e.g. $C_{21} = q_{21}n$

We define the critical density, $n_{\rm cr} = A_{21}/q_{21}$, as the particle density at which C_{21} is equal to A_{21}

In the high-density limit, $n >> n_{\rm cr}$, the collisional terms dominate and

$$\frac{n_2}{n_1} = \frac{C_{12} + B_{12}\bar{J}}{C_{21} + A_{21} + B_{21}\bar{J}} \sim \frac{C_{12}}{C_{21}} = \frac{g_2}{g_1}\exp(-h\nu/kT_{\rm kin})$$ which implies $T_{\rm ex} = T_{\rm kin}$

In the low-density limit, $n << n_{\rm cr}$, the radiative terms dominate and

$$\frac{n_2}{n_1} = \frac{C_{12} + B_{12}\bar{J}}{C_{21} + A_{21} + B_{21}\bar{J}} \sim \frac{B_{12}\bar{J}}{A_{21} + B_{21}\bar{J}} = \frac{g_2}{g_1}\exp(-h\nu/kT_{\rm rad})$$ which implies $T_{\rm ex} = T_{\rm rad}$

You'll work this out more fully in the homework

Maser emission

Under some conditions actually attained in the interstellar gas, certain transitions of specific molecules can have a "population inversion" in which

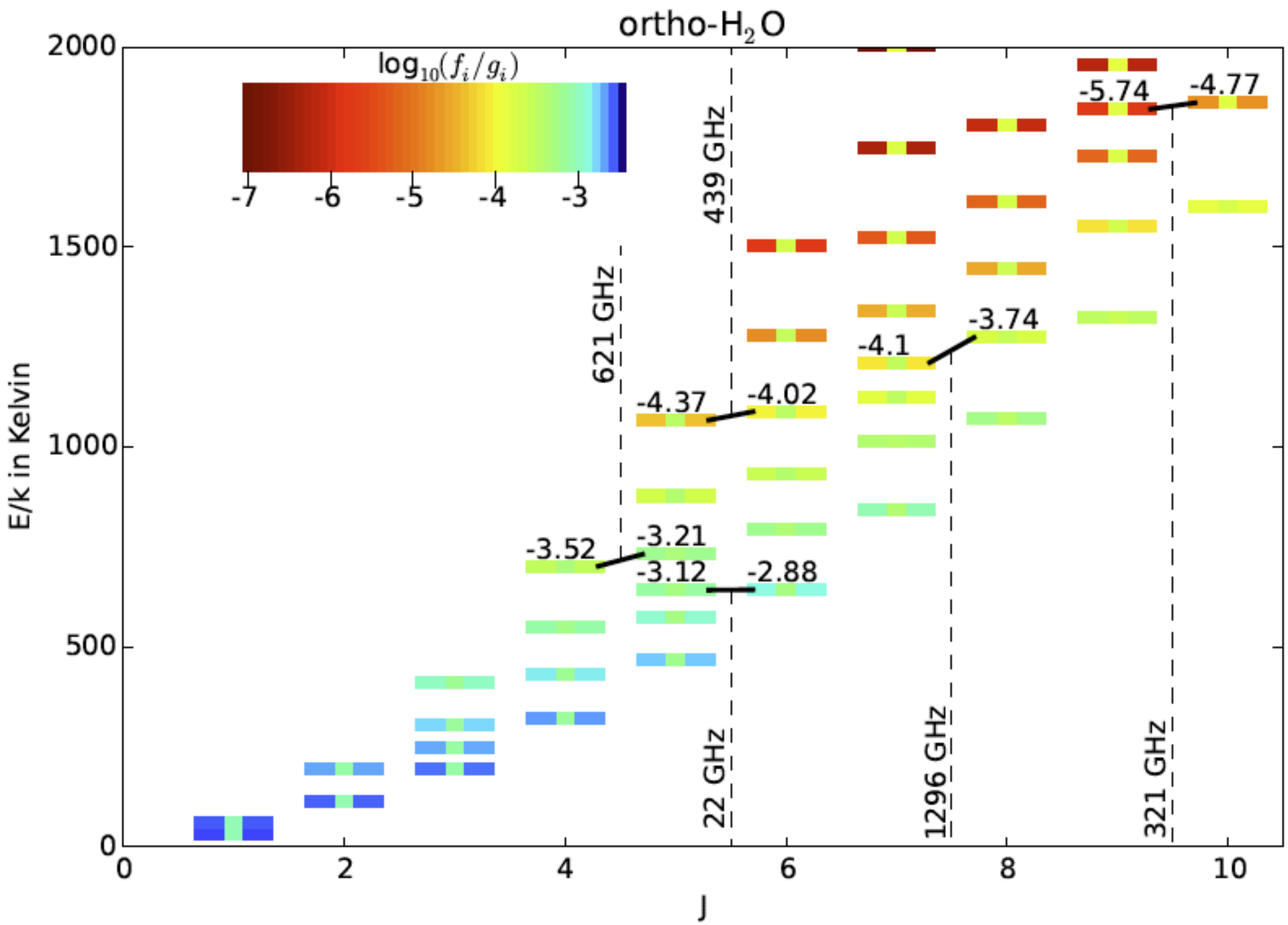
$$\frac{n_u}{n_l} > \frac{g_u}{g_l}$$

This implies a negative excitation temperature and a negative absorption coefficient $\frac{h\nu}{4\pi}[n_l\, B_{lu} \; - n_u\, B_{ul}\;]\, \phi(\nu)$

Leading to the exponential amplification of radiation in accord with $\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu\, I_\nu$

Maser emission

Most notable example: the 22 GHz transition of H_2O



Neufeld et al. (2021) ApJ

Maser emission

Most notable example: the 22 GHz transition of H_2O

In warm environments ($T_{\mathrm{kin}} > 300 - 400$ K), we see small spots of maser radiation with brightness temperatures T_{B} up to 10^{14} K

A fascinating phenomenon in its own right, but also a fantastic "tool", because emission that bright can be observed using the techniques of *radio interferometry*

VLBI (Very Long Baseline Interferometry)

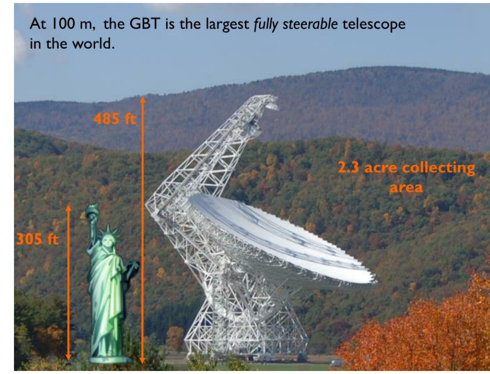
100 m single dish (e.g. GBT)

Angular resolution is approximately

$\theta \sim \lambda/D$ = 1.4 x 10^{-4} rad = $28''$

(only slightly better than human eye, for which $\theta \sim 40''$)

R. Prestage, NRAO Green Bank



Interferometer (e.g. VLA)

$\theta \sim \lambda/D_{max}$ = 4.3 x 10^{-7} rad = $0.09''$

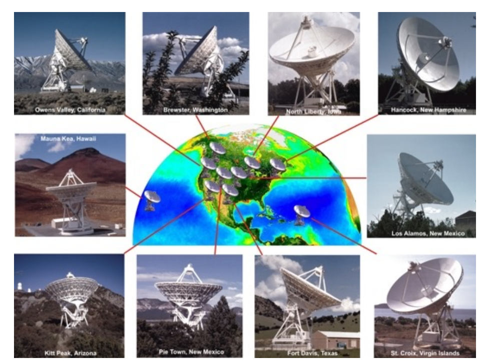
Maximum separation (*not* individual dish size) = 36 km

(only slightly better than HST, for which $\theta \sim 0.05''$)

Very Long Baseline Interferometer

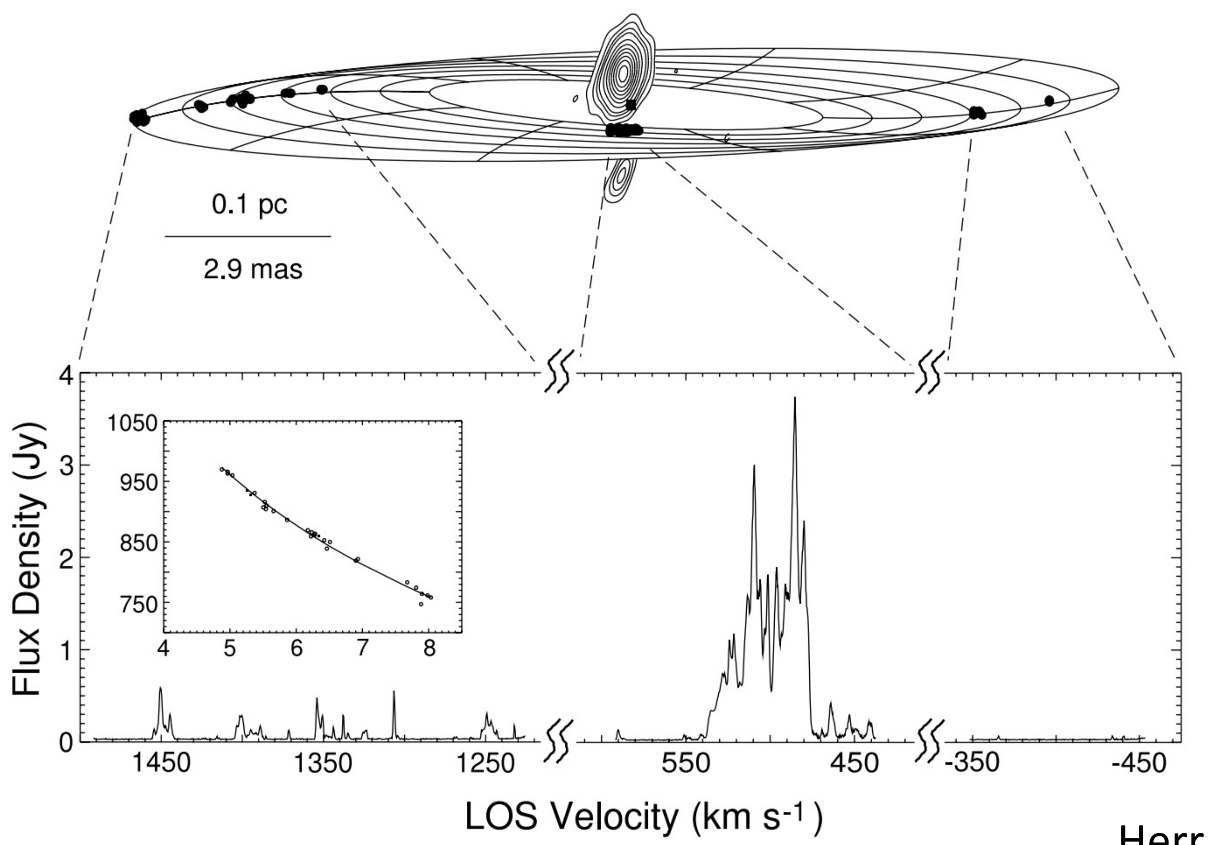
$\theta \sim \lambda/D_{max}$ = 1.4 x 10^{-9} rad = $0.00028''$ = 280 μas

Maximum separation (*not* individual dish size) up to 10,000 km



Maser emission: example application

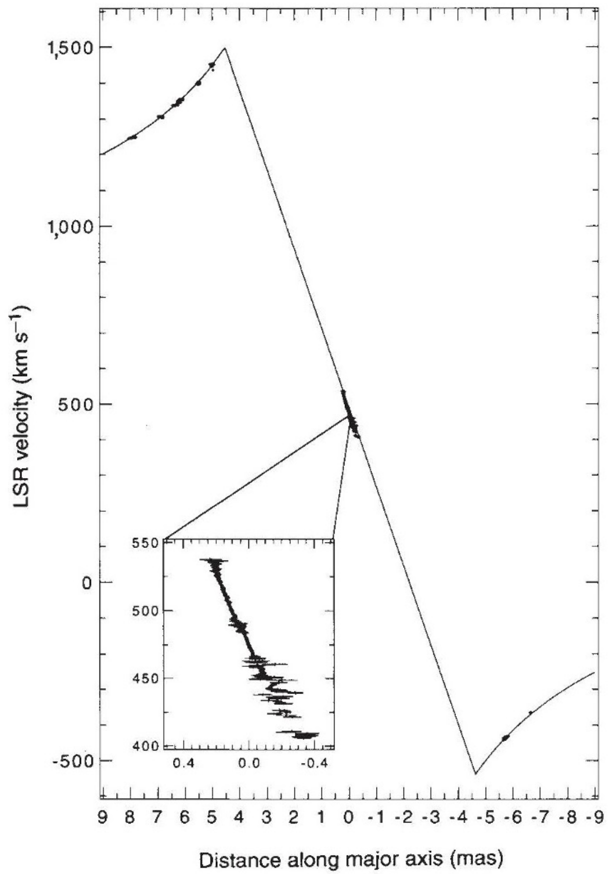
VLBI observations of the 22 GHz water maser towards the active galaxy NGC 4258 reveal a warped circumnuclear disk viewed nearly edge-on

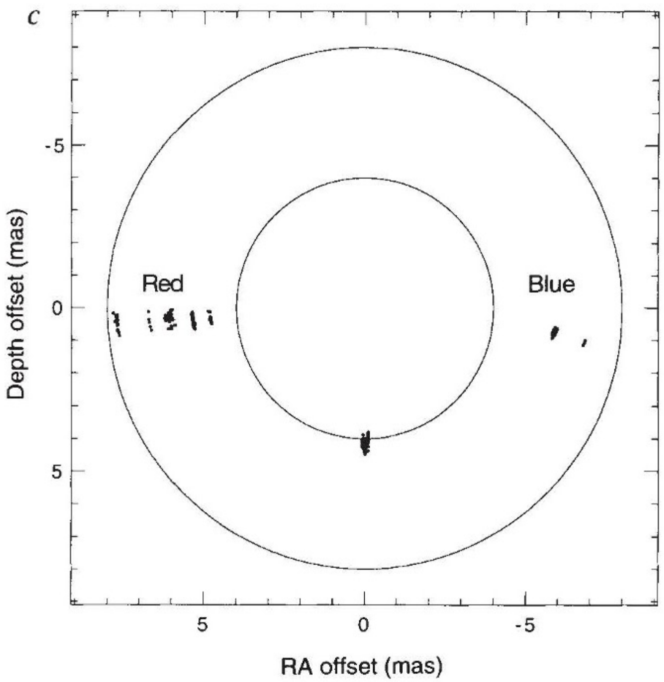


Herrnstein et al. 1999

Maser emission: example application

Each maser spot is tagged with its Doppler velocity along the line-of-sight, revealing a disk in Keplerian rotation



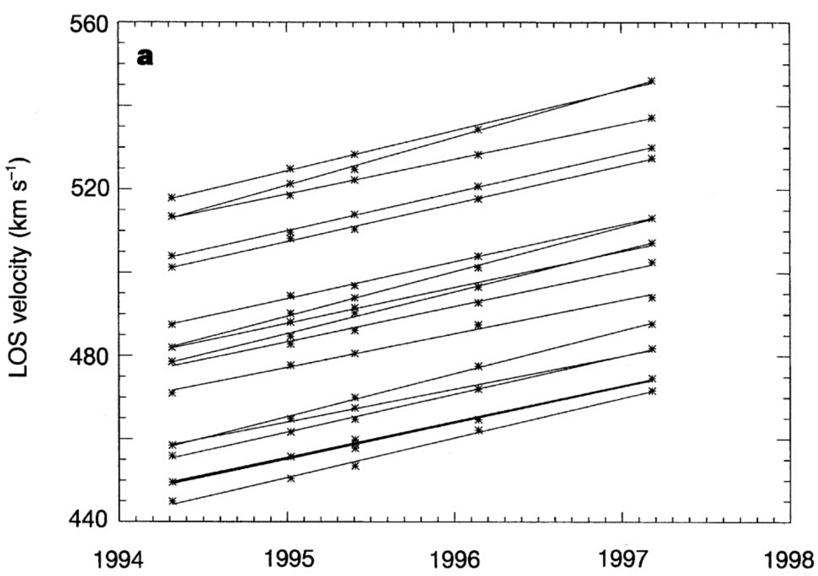


Central mass = $3.6 \times 10^7\ M_\odot$ within 0.13 pc
➔ black hole, not star cluster

Miyoshi et al. 1993

Maser emission: example application

Over a period of years, the Doppler motions associated with the radially beamed spots are observed to march redward



Herrnstein et al. 1999

Acceleration, $a = v^2/r$ = 9.5 km/s per yr

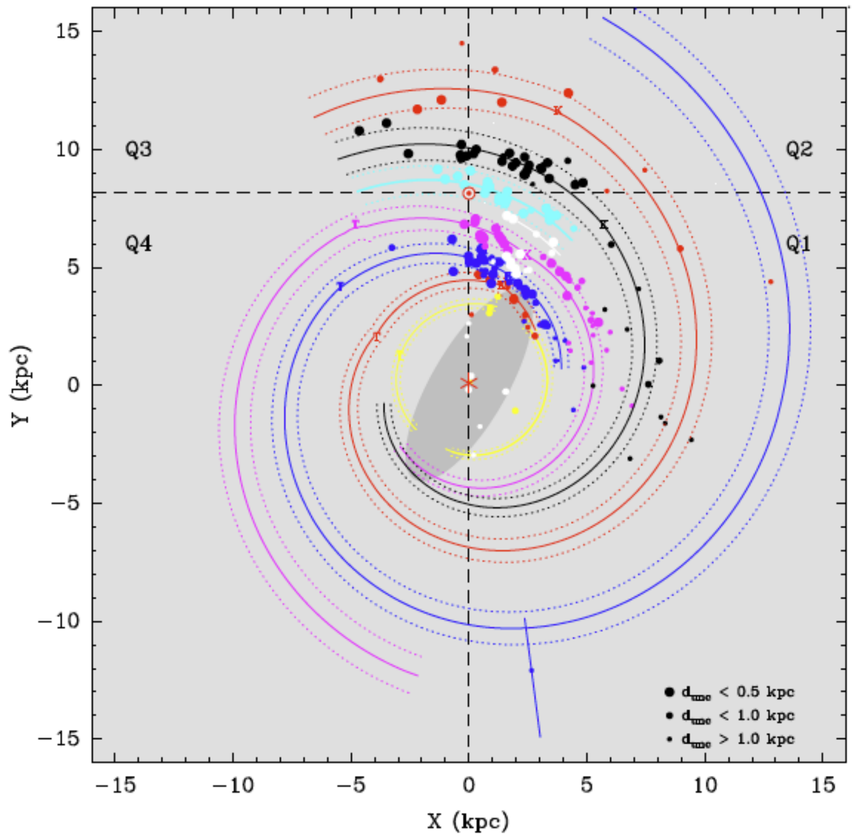
They measured a and v, so they could determine r

This defines that actual physical scale of the system (in pc), not the angular size

We have a ***standard ruler*** for something that is spatially-resolved, so we have a distance indicator, $d = r/\theta$

Latest determination (Pesce et al 2020):
d = 7.58 ± 0.11 Mpc (1.5% uncertainty)

Other applications include parallax measurements



Distance to Galactic Center

R_0 = 8150 ± 150 pc

(but not as good as a determination using IR interferometry of stars)

$R_0 = 8178 \pm 13_{stat} \pm 22_{sys}$ pc

from the GRAVITY collaboration (2019)

Structure of the Milky Way, based on trigonometric parallaxes from water and methanol masers in regions of star formation (Reid et al. 2019: BeSSel and VERA surveys)

Lecture 6
Scattering

Goals: understand

Scattering

Radiative transfer with scattering

The radiative diffusion approximation

READING: *R&L* 1.6, 1.7

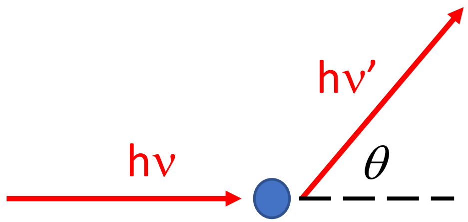
Scattering

There's another important radiative process that we have not yet considered: the *scattering* of radiation

Here, photons are redirected but neither absorbed or emitted

Key example: scattering by electrons (a.k.a. Thomson scattering)

As we'll see later, if $h\nu << m_e c^2$, the scattering is coherent in the electron rest frame with $\nu' = \nu$



Electron scattering is not exactly isotropic (we'll see that later too), but has a forward-backward symmetry and can be approximated as isotropic

Scattering

To account for scattering that is coherent and isotropic, we can just add two terms to the transfer equation for a given ray:

$$\frac{dI_\upsilon}{ds} = j_\upsilon + \sigma_\nu J_\upsilon - \alpha_\upsilon \ I_\upsilon - \sigma_\nu I_\upsilon$$

where σ_ν is a scattering coefficient with dimensions length^{-1} (just like the absorption coefficient α_ν). The mean distance travelled by a photon before is scattered is $1/\sigma_\nu$.

Annoying aside: *R&L* and most other texts use the same symbol for the scattering coefficient as for the cross-section.

$\sigma_\nu I_\upsilon$ is the rate (per unit distance along the ray) at which intensity is removed from this ray by scattering out of this direction

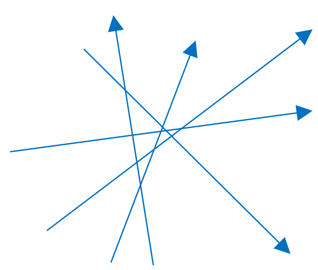
$\sigma_\nu J_\upsilon$ is the rate (per unit distance along the ray) at which energy is added to this ray by scattering of radiation originally moving in other directions

Scattering

To account for scattering that is coherent and isotropic, we can just add two terms to the transfer equation for a given ray:

$$\frac{dI_\upsilon}{ds} = j_\upsilon + \sigma_\nu J_\upsilon - \alpha_\upsilon \, I_\upsilon - \sigma_\nu \, I_\upsilon$$

This change to our equation is deceptively simple. In reality, it complicates the situation greatly by *coupling* the radiative transfer equations we solve for different rays



Without scattering I could solve the transfer equation separately for each ray
With scattering, I have to solve for all rays simultaneously

Scattering in LTE

To keep things (relatively) simple, let's assume that we are in LTE with $j_\nu = \alpha_\nu B_\nu(T)$

$$\frac{dI_\nu}{ds} = \alpha_\nu \, B_\nu + \sigma_\nu \, J_\nu - \alpha_\nu \, I_\nu - \sigma_\nu \, I_\nu$$

We can extend our definition of optical depth by writing $d\tau_\nu \equiv (\alpha_\nu + \sigma_\nu)\, ds$ to obtain

$$\frac{dI_\nu}{d\tau_\nu} = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu} B_\nu + \frac{\sigma_\nu}{\alpha_\nu + \sigma_\nu} J_\nu - I_\nu = S_\nu - I_\nu$$

With the inclusion of scattering, our source function becomes $S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu)\, J_\nu$

where $\epsilon_\nu \equiv \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu}$

Physical meaning

Photons can interact with matter by being absorbed or getting scattered

The mean distance between interaction events is

$$l_{\mathrm{mfp}} = \frac{1}{\alpha_\nu + \sigma_\nu}$$

In any such interaction, the probability of absorption is $\epsilon_\upsilon = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu}$

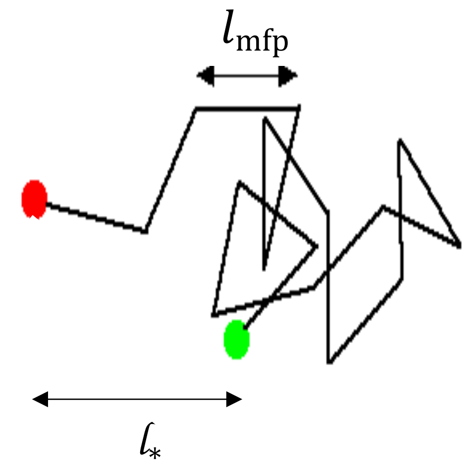
and the probability of scattering is $1 - \epsilon_\upsilon = \frac{\sigma_\nu}{\alpha_\nu + \sigma_\nu}$

$1 - \epsilon_\upsilon$ is called the single-scattering *albedo* (recall the definition of planetary albedos)

Random walk

In the presence of scattering, photons do a random walk where the mean step size is l_{mfp}

Q1 (poll, individually) On average, how many steps will a photon take before it is absorbed?

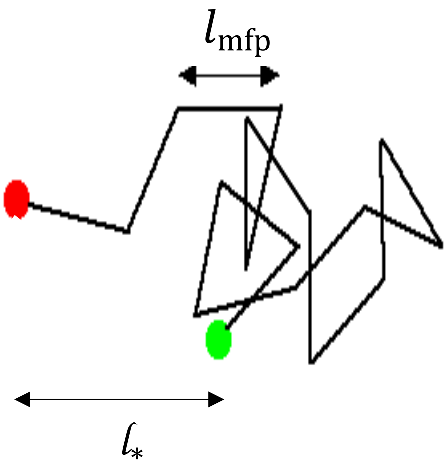


Choices: a,b,c,d,e

a) $1/\epsilon_\nu{}^2$
b) $1/\epsilon_\nu$
c) $1/(1-\epsilon_\nu)$
d) ϵ_ν
e) $(1-\epsilon_\nu)$

Random walk

In the presence of scattering, photons do a random walk where the mean step size is l_{mfp}

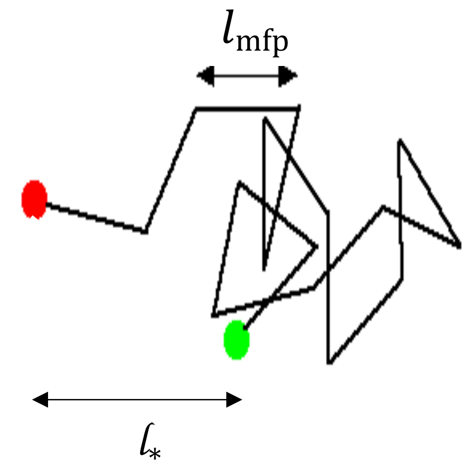


Q1 (poll, individually) On average, how many steps will a photon take before it is absorbed?

A The average number of steps, *N,* is $1/\epsilon_\nu$

Q2 (poll, in groups) After N steps, what is the r.m.s. displacement in the x-direction (or any other direction)?

Random walk



In the presence of scattering, photons do a random walk where the mean step size is l_{mfp}

Q1 (poll, individually) On average, how many steps will a photon take before it is absorbed?

A The average number of steps, *N*, is $1/\epsilon_\nu$

Q2 (poll, in groups) After N steps, what is the r.m.s. displacement in the x-direction (or any other direction)?

After *N* random steps, the mean square displacement is

$$\langle(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\rangle = N {l_{\mathrm{mfp}}}^2$$

The mean distance travelled (r.m.s. displacement) is

$$N^{1/2} l_{\mathrm{mfp}} = \left(\frac{\alpha_\nu + \sigma_\nu}{\alpha_\nu}\right)^{1/2} \frac{1}{\alpha_\nu + \sigma_\nu} = \left(\frac{1}{\alpha_\nu(\alpha_\nu + \sigma_\nu)}\right)^{1/2}$$

In any one direction (e.g. the x-direction) it is $l_* = \langle(\Delta x)^2\rangle^{1/2} = \left(\frac{1}{3\alpha_\nu(\alpha_\nu + \sigma_\nu)}\right)^{1/2}$

Angular moments of the transfer equation

Previously, we noted that the radiative transfer equation for a single ray (without scattering)

$$\frac{dI_v}{ds} = j_v - \alpha_v \, I_v$$

could be written for all rays at once

$$\widehat{\boldsymbol{k}}\,.\boldsymbol{\nabla} I_v(\widehat{\boldsymbol{k}}) = \; j_v - \alpha_v \, I_v(\widehat{\boldsymbol{k}})$$

Taking the zeroth and first moments, we got

$\boldsymbol{\nabla}.\boldsymbol{F_v} = 4\pi j_v - 4\pi\alpha_v \, J_v$ Energy conservation

$4\pi \, \boldsymbol{\nabla}.\boldsymbol{K}_v = -\alpha_v \;\; \boldsymbol{F_v}$ Momentum conservation

Now let’s add scattering

Angular moments of the transfer equation

Our transfer equation becomes

$$\frac{dI_\upsilon}{ds} = j_\upsilon + \sigma_\nu\ J_\upsilon - \alpha_\upsilon\ I_\upsilon - \sigma_\nu\ I_\upsilon$$

could be written for all rays at once

$$\hat{\boldsymbol{k}}\,.\boldsymbol{\nabla} I_\upsilon(\hat{\boldsymbol{k}}) = \ j_\upsilon + \sigma_\nu\ J_\upsilon - \alpha_\upsilon\ I_\upsilon(\hat{\boldsymbol{k}}) - \sigma_\nu I_\upsilon(\hat{\boldsymbol{k}})$$

Taking the zeroth and first moments, we got

$$\boldsymbol{\nabla}.\boldsymbol{F_\upsilon} = 4\pi j_\upsilon + 4\pi\sigma_\nu J_\upsilon - 4\pi\alpha_\upsilon J_\upsilon - 4\pi\sigma_\nu J_\upsilon\ = 4\pi j_\upsilon - 4\pi\alpha_\upsilon J_\upsilon$$

Energy conservation unchanged (scattering conserves photons)

$$4\pi\ \boldsymbol{\nabla}.\boldsymbol{K}_\upsilon = -(\alpha_\upsilon\ + \sigma_\nu)\ \boldsymbol{F_\upsilon}$$

Momentum conservation modified because of additional momentum transfer to gas

Plane parallel geometry

Now let's suppose we have plane-parallel geometry, again in LTE, with the z-axis along the direction where the intensity changes

In other words, we are assuming $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$

And the only non-zero components of ***F*** and ***K*** are F_z and K_{zz}

The moment equations become

$$\frac{dF_{\upsilon z}}{dz} = 4\pi j_\upsilon - 4\pi\alpha_\upsilon J_\upsilon = -4\pi\alpha_\upsilon(J_\upsilon - B_\upsilon)$$

$$\frac{dK_{\upsilon zz}}{dz} = -(\alpha_\upsilon + \sigma_\upsilon)F_{\upsilon z}$$

Plane parallel geometry

The moment equations relate the derivative of one moment to the value of the previous one.

$$\frac{dF_{\upsilon z}}{dz} = 4\pi j_\upsilon - 4\pi\alpha_\upsilon J_\upsilon = -4\pi\alpha_\upsilon(J_\upsilon - B_\upsilon)$$

$$\frac{dK_{\upsilon zz}}{dz} = -(\alpha_\upsilon + \sigma_\upsilon)F_{\upsilon z}/4\pi$$

To "close" the system of equations, we need something else

For isotropic radiation, we found previously that $K_{\upsilon zz}$= $\frac{1}{3}J_\upsilon$, and we'll see that this turns out to be a *reasonable* approximation more generally

If we make this "Eddington approximation," we can derive a 2nd order ODE for J_υ (differentiating the second equation again and substituting for dF_υ/dz to obtain)

$$\frac{d^2 J_\upsilon}{dz^2} \sim 3\,\frac{d^2 K_{\upsilon zz}}{dz^2} = 3(\alpha_\upsilon + \sigma_\upsilon)\alpha_\upsilon(J_\upsilon - B_\upsilon)$$

Eddington approximation

The Eddington approximation is generally good when the radiation is nearly-isotropic as in stellar interiors

It is also *exactly* true in two special cases

1) For "semi-isotropic radiation" (travelling in one hemisphere),
i.e. if $I_\upsilon = a$ for $\mu > 0$
$I_\upsilon = 0$ for $\mu < 0$ where $\mu = \cos\theta$ as before

2) When I_υ is a linear function of μ, $I_\upsilon = a + b\mu$,

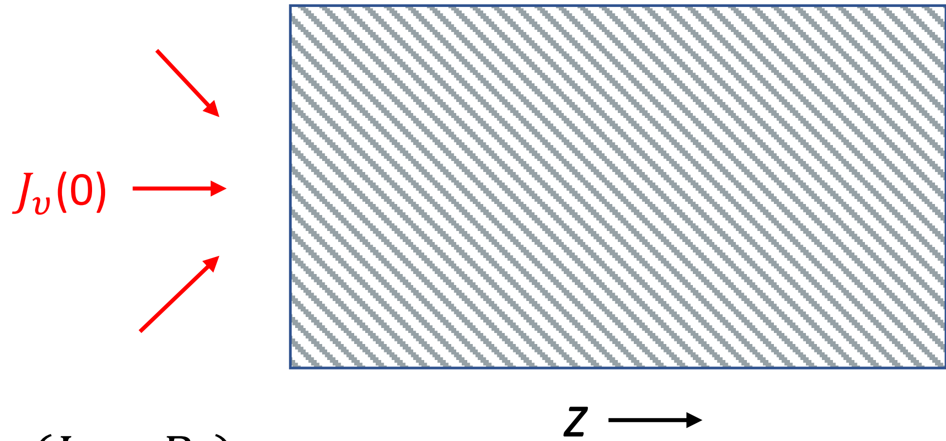
for which $J_\upsilon = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu \, d\mu = \frac{1}{2} \int_{-1}^{1} (a + b\mu) \; d\mu = a + 0b$

and $K_{\upsilon zz} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu \, \mu^2 \, d\mu = \frac{1}{2} \int_{-1}^{1} (a\mu^2 + b\mu^3) \; d\mu = \frac{1}{3} a + 0b$

Application to an irradiated semi-infinite slab

Let's solve our second order ODE for the case of a "semi-infinite," isothermal slab of material irradiated by semi-isotropic radiation at its surface

$$\frac{d^2 J_\upsilon}{dz^2} = 3(\alpha_\upsilon + \sigma_\upsilon)\alpha_\upsilon(J_\upsilon - B_\upsilon)$$



We can introduce a special optical depth, $\tau_* = dz / \ell_*$, and rewrite this

$$\frac{d^2 J_\upsilon}{d\tau_*{}^2} = (J_\upsilon - B_\upsilon)$$

τ_* measures z distance in units of the mean distance between absorption events

The solution is $(J_\upsilon - B_\upsilon) = a e^{-\tau_*} + b e^{+\tau_*}$

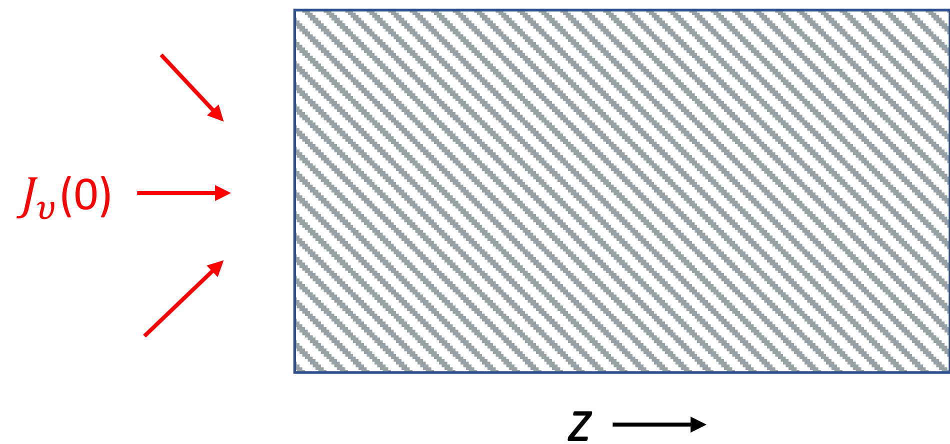
Boundary conditions

1) finite J_υ at large τ_* ➔ $b = 0$
2) $J_\upsilon = J_\upsilon(0)$ at the irradiated surface

Application to an irradiated semi-infinite slab

Hence $(J_\nu - B_\nu) = (J_\nu(0) - B_\nu)e^{-\tau*}$

➔ $J_\nu(\tau_*) = J_\nu(0)\, e^{-\tau*} + B_\nu\,(1 - e^{-\tau*})$



Near the surface (τ_* << 1 or equivalently z << ℓ_*), J_ν is determined by the incident radiation, $J_\nu \sim J_\nu(0)$

In the interior (τ_* >> 1 or equivalently z >> ℓ_*), J_ν reaches thermal equilibrium with the matter $J_\nu \sim B_\nu$

Another relaxation equation. The distance ℓ_* is often termed the thermalization length. In the homework, you'll solve a similar problem but with a source of luminosity at the center of a finite slab

Lecture 7
Review of EM theory

Goals:

Understand the radiative diffusion approximation

Review:

Lorentz force
Maxwell's Equations
EM potentials

READING: *R&L* 2.1,2.5,2.3

The radiative diffusion (Rosseland) approximation

If we are in the deep interior of a medium where the temperature changes slowly on the scale of the thermalization length
(i.e. if $dT/dz << T/\ell_*$),

we can approximate J_υ very accurately by B_υ (T)

Moreover, the radiation is very nearly isotropic, so that $J_\upsilon = 3K_\upsilon$

These are extraordinarily good approximations in stellar interiors

We can then compute the flux, using the first moment of the transfer equation

$$\frac{dK_\upsilon}{dz} = -(\alpha_\upsilon + \sigma_\upsilon)F_\upsilon/4\pi$$

The radiative diffusion (Rosseland) approximation

$$F_\upsilon = -\frac{4\pi}{\sigma_\upsilon + \alpha_\upsilon}\frac{dK_\upsilon}{dz} = -\frac{4\pi}{3(\sigma_\upsilon + \alpha_\upsilon)}\frac{dJ_\upsilon}{dz}$$

$$= -\frac{4\pi}{3(\sigma_\upsilon + \alpha_\upsilon)}\frac{dB_\upsilon(T)}{dz} = -\frac{4\pi}{3(\sigma_\upsilon + \alpha_\upsilon)}\frac{\partial B_\upsilon}{\partial T}\frac{dT}{dz}$$

The energy flux is proportional to the temperature gradient, as we might have expected

To determine the total (frequency-integrated flux), we can write

$$F = -\frac{4\pi}{3\alpha_R}\frac{dB(T)}{dz} = -\frac{c}{3\alpha_R}\frac{\partial u}{\partial T}\frac{dT}{dz} =- -\frac{4acT^3}{3\alpha_R}\frac{dT}{dz}$$

where α_R is the average value of $\sigma_\upsilon + \alpha_\upsilon$ and using $u = 4\pi B/c = aT^4$

The Rosseland mean "opacity"

For a "grey" medium with $\sigma_\upsilon + \alpha_\upsilon$ independent of υ, the Rosseland mean opacity α_R is simply $\sigma_\upsilon + \alpha_\upsilon$

In general, the appropriate average is a weighted harmonic mean

$$\frac{1}{\alpha_R} = \frac{\int (\sigma_\upsilon + \alpha_\upsilon)^{-1} \frac{\partial B_\upsilon}{\partial T} d\upsilon}{\int \frac{\partial B_\upsilon}{\partial T} d\upsilon}$$

The integral in the numerator is dominated by the frequencies where $(\sigma_\upsilon + \alpha_\upsilon)$ is smallest. This is where the flux is transported most rapidly

The radiative diffusion equation

This is called the radiative diffusion (or Rosseland) equation

$$F = -\frac{4\pi}{3\alpha_R}\frac{B(T)}{dz} = -\frac{c}{3\alpha_R}\frac{du(T)}{dz} = -\frac{c}{3\alpha_R}\frac{\partial u}{\partial T}\frac{dT}{dz} = -\frac{4acT^3}{3\alpha_R}\frac{dT}{dz}$$

It has the classic form of a diffusion equation for some quantity *Q* (in this case energy)

Flux of *Q* = diffusion coefficient x gradient in the density of *Q*

And the diffusion coefficient is always of order

Speed of the carrier of *Q* x mean distance travelled

Definitions of the electric and magnetic fields

Operational definition of ***E***, ***B*** and *q*:

Lorentz force on a charged particle, $\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c)$

→ Rate of work done on particle = $\boldsymbol{v} \cdot \boldsymbol{F} = q\, \boldsymbol{v} \cdot \boldsymbol{E}$

Continuum description, $\boldsymbol{f} = \rho\, \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}/c$

Force per unit volume

Current density

(flux of charge)

Charge density

Rate of work done per unit volume = $\boldsymbol{j} \cdot \boldsymbol{E}$

Note on units

R&L use Gaussian-c.g.s units, which are widely used in theoretical physics and astronomy

- In this system, unlike in the SI system, there are no dimensional constants, μ_0 and ϵ_0, and the speed of light appears explicitly in Maxwell's equations.
- Coulomb's Law becomes $F = q_1 q_2 / r^2$
- The unit of charge is the statCoulomb (1 statC = 1 cm$^{3/2}$ g$^{1/2}$ s^{-1})

    also known at the electrostatic unit (esu) or (rarely) the Franklin

  The electronic charge, e = 4.803204 x 10^{-10} statC
- The unit of magnetic field is the Gauss (1 G = cm$^{-1/2}$ g$^{1/2}$ s^{-1} = 10^{-4} T)
- Lorentz force on a charged particle, $\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c)$

    so *E* and *B* have the same unit

Maxwell's Equations

Gauss's Law: $\boldsymbol{\nabla}.\boldsymbol{E} = 4\pi\rho$

No magnetic monopoles: $\boldsymbol{\nabla}.\boldsymbol{B} = 0$

Faraday's Law: $\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{dt}$

Ampere's Law: $\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c}\frac{\partial\boldsymbol{E}}{dt} + \frac{4\pi}{c}\boldsymbol{j}$

Implications of Maxwell's equations

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{dt} + \frac{4\pi}{c}\boldsymbol{j}$$

$$\frac{c}{4\pi}\boldsymbol{\nabla}.$$

$$\frac{c}{4\pi}\boldsymbol{E}.$$

$$\boldsymbol{0} = \frac{\partial \boldsymbol{\rho}}{dt} + \boldsymbol{\nabla}.\boldsymbol{j}$$

Conservation of charge

$$\frac{c}{4\pi}\boldsymbol{E}.\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{8\pi}\frac{\partial \boldsymbol{E}^2}{dt} + \boldsymbol{j}.\boldsymbol{E}$$

$$\frac{c}{4\pi}\boldsymbol{B}.\boldsymbol{\nabla}\times\boldsymbol{E} - \frac{c}{4\pi}\boldsymbol{\nabla}.(\boldsymbol{E}\times\boldsymbol{B}) = \frac{\boldsymbol{E}}{4\pi}.\frac{\partial \boldsymbol{E}}{dt} + \boldsymbol{j}.\boldsymbol{E}$$

$$-\frac{B}{4\pi}.\frac{\partial \boldsymbol{B}}{dt} - \frac{c}{4\pi}\boldsymbol{\nabla}.(\boldsymbol{E}\times\boldsymbol{B}) = \frac{1}{8\pi}\frac{\partial \boldsymbol{E}^2}{dt} + \boldsymbol{j}.\boldsymbol{E}$$

$$0 = \frac{\partial}{\partial t}\left(\frac{\boldsymbol{E}^2 + B^2}{8\pi}\right) + \boldsymbol{\nabla}.\left(\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}\right) + \boldsymbol{j}.\boldsymbol{E}$$

Implications of Maxwell's equations

$$0 = \frac{\partial}{dt}\left(\frac{\boldsymbol{E}^2 + B^2}{8\pi}\right) + \boldsymbol{\nabla}.\left(\frac{\boldsymbol{c}}{4\pi}\boldsymbol{E}\times\boldsymbol{B}\right) + \boldsymbol{j}\,.\,\boldsymbol{E}$$

Energy density, *u*

Energy flux, *F*

(Poynting vector)

Power dissipation

(per unit volume)

Conservation of energy

Electromagnetic potentials

Making use of the vector identities

$\boldsymbol{\nabla}.(\boldsymbol{\nabla} \times \boldsymbol{V}) = 0$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = \mathbf{0}$

we can automatically enforce $\boldsymbol{\nabla}.\,\boldsymbol{B} = 0$ by writing

$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

and automatically enforce Faraday's Law

$$0 = \boldsymbol{\nabla}\times\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{dt} = \boldsymbol{\nabla}\times\left(\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{dt}\right)$$

by writing

$$\left(\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{dt}\right) = -\boldsymbol{\nabla}\phi$$

Gauge transformation

When we write

$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ $\quad\quad$ $\left(\boldsymbol{E} + \frac{1}{c}\frac{\partial A}{dt}\right) = -\boldsymbol{\nabla}\phi$

We have some flexibility in choosing $\boldsymbol{A}$ and ϕ

In particular, because $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0$,

we can add the gradient of any scalar function ψ to A, provided we also subtract $(1/c)d\psi/dt$ from ϕ

This "gauge transformation" leaves *E* and *B* unchanged

$A \rightarrow A + \boldsymbol{\nabla}\psi$

$\phi \rightarrow \phi - (1/c)\partial\psi/\partial t$

Lecture 8
EM waves and polarization

Goals: understand

Maxwell's equations in the Lorentz gauge

Polarization: astrophysical context

READING: *R&L* 2.4

Lorentz gauge

For a suitable choice of $\psi(\boldsymbol{x},t)$, we can always arrange things so that $\boldsymbol{\nabla}.\boldsymbol{A} = -(1/c)\partial\phi/\partial t$

This is called the *Lorentz gauge*

With this choice, the two remaining Maxwell's equations become

$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t} = -4\pi\rho$ Gauss's Law

$\boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial^2 t} = -4\pi\boldsymbol{j}/\mathrm{c}$ Ampere's Law

Wave solution

$$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t} = -4\pi\rho$$ Gauss' Law

$$\boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial^2 t} = -4\pi\boldsymbol{j}/\mathrm{c}$$ Ampere's Law

In a vacuum, ρ = 0 and $\boldsymbol{j}$ = 0 and the solution is

$$\phi = \phi_0 e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)}$$

$$\boldsymbol{A} = \boldsymbol{A_0} e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)}$$

As usual, ϕ_0 and $\boldsymbol{A_0}$ are complex, with the argument representing phase, and we take the real part of the RHS

Wave solution: relation between $\boldsymbol{A_0}$ and ϕ_0

$$\phi = \phi_0 e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)}$$

$$\boldsymbol{A} = \boldsymbol{A_0} e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)}$$

This solution is premised on the Lorentz Gauge, which relates $\boldsymbol{A_0}$ to ϕ_0

$\boldsymbol{\nabla}.\boldsymbol{A} = -(1/c)\partial\phi/\partial t$ ➔ $i\boldsymbol{k}.\boldsymbol{A_0} = -(1/c)(-ikc\phi_0)$

$\phi_0 = \hat{\boldsymbol{k}} \,.\, \boldsymbol{A_0}$ is the projection of $\boldsymbol{A_0}$ onto the direction of propagation

Wave solution: *E* and B fields

$$\boldsymbol{A} = \boldsymbol{A_0} e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)}$$

$$\Rightarrow \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \underbrace{(i\boldsymbol{A_0} \times \boldsymbol{k})}_{\boldsymbol{B_0}}\, e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)}$$

$$\phi = \phi_0 e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)}$$

$$\Rightarrow \quad \boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial A}{dt} = \underbrace{(-i\boldsymbol{k}\phi_0 + ik\boldsymbol{A_0})}_{\boldsymbol{E_0}}\, e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)}$$

Note that $\boldsymbol{E} \times \widehat{\boldsymbol{k}} = (0 + ik\boldsymbol{A_0} \times \widehat{\boldsymbol{k}})\, e^{i(\boldsymbol{k}.\boldsymbol{x} - kct)} = \boldsymbol{B}$

➔ ***B*** and ***E*** are mutually perpendicular and vary in phase

Wave solution: *E* and B fields

Waves are transverse

$$\boldsymbol{B_0}.\boldsymbol{k} = \ i\boldsymbol{A_0} \times \boldsymbol{k}\ .\ \boldsymbol{k} = 0$$

$$\boldsymbol{E_0}.\boldsymbol{k} = \ i(\boldsymbol{A_0}.\widehat{\boldsymbol{k}} - \phi_0)\, k^2 = 0$$

Thus, $\boldsymbol{B} = \boldsymbol{E} \times \widehat{\boldsymbol{k}}$ has the same magnitude as $\boldsymbol{E}$

The flux is $\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{c}{4\pi}\, E(t)B(t)\,\widehat{\boldsymbol{k}} = \frac{c}{4\pi}\, E^2(t)\,\widehat{\boldsymbol{k}} = \frac{c}{4\pi}\, B^2(t)\,\widehat{\boldsymbol{k}}$

Averaged over one cycle, we get

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi}\boldsymbol{E_0}.\boldsymbol{E_0}^*\,\widehat{\boldsymbol{k}} \ = \frac{c}{8\pi}\boldsymbol{B_0}.\boldsymbol{B_0}^*\,\widehat{\boldsymbol{k}}$$

Spectral analysis

Let's now consider the Fourier transform of *E(t),* computed over some period T

$\tilde{E}_T(\omega) = \frac{1}{2\pi}\int_0^T e^{i\omega t}\, E(t)dt$ (complex)

Parseval's theorem tells us that

$$\int_0^T E^2(t)dt = 4\pi \int_0^\infty |\tilde{E}_T(\omega)|^2\, d\omega$$

But $\int_0^T E^2(t)dt = \frac{4\pi T}{c}\langle \boldsymbol{S}\rangle_T$, so the average (frequency-integrated) flux over this period is

$\langle \boldsymbol{S}\rangle_T = \frac{c}{T}\int_0^\infty |\tilde{E}_T(\omega)|^2\, d\omega$

Spectral analysis

$$\langle S \rangle_T = \frac{c}{T} \int_0^\infty |\tilde{E}_T(\omega)|^2 \, d\omega$$

is an expression for the total flux

$$F \ = \int_0^\infty F_\upsilon \, d\upsilon = \int_0^\infty (F_\upsilon / 2\pi) \, d\omega$$

Hence, we may equate the integrands in these two equations and obtain

$$F_\upsilon = \frac{2\pi c}{T} \ |\tilde{E}_T(\omega)|^2$$

Polarization: astrophysical context

Polarization provides key astrophysical information

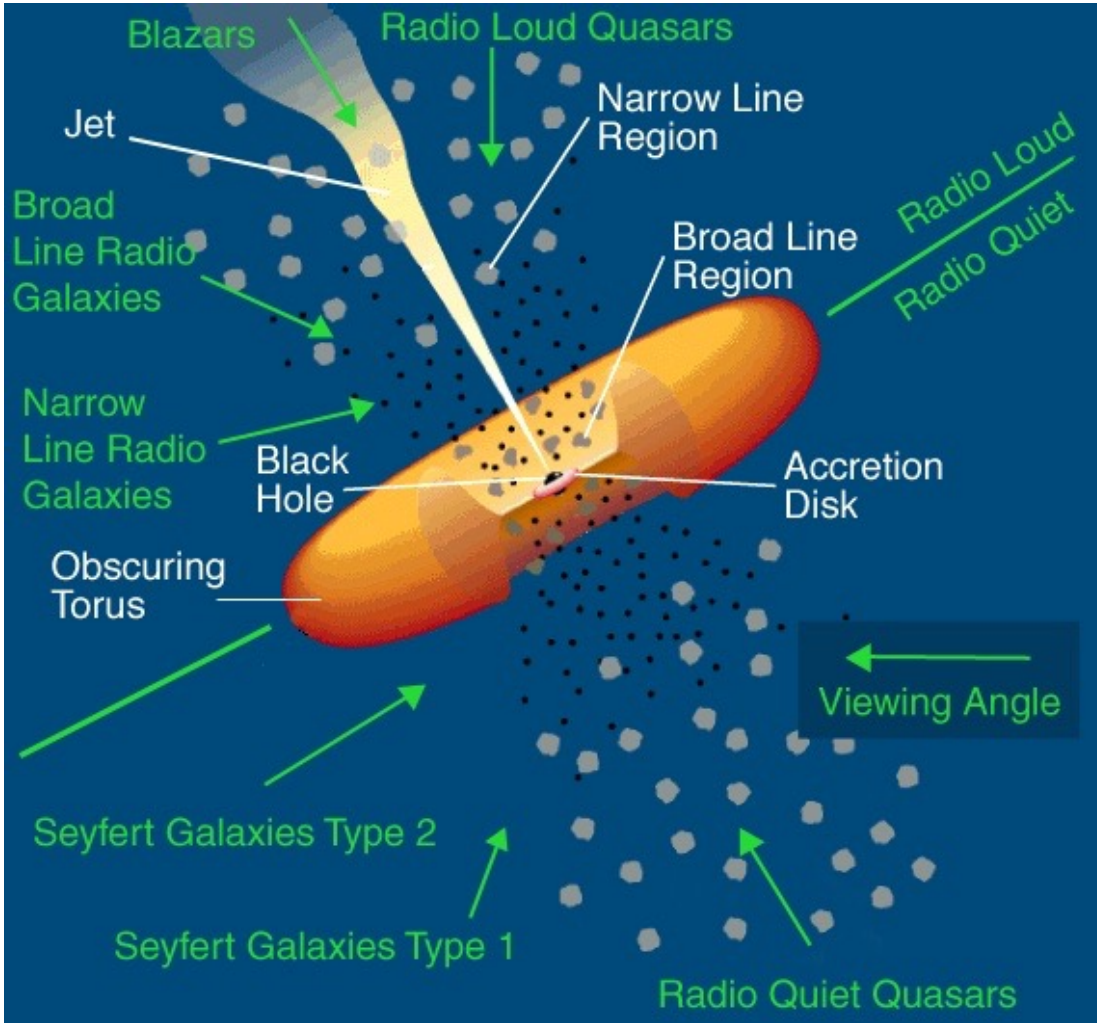
Examples:

Scattered radiation and the “unified AGN model”

Probing μG magnetic fields with

- Polarized synchrotron radiation
- Polarized dust emission
- Starlight transmitted through the ISM
- Faraday rotation within a magnetized plasma

Polarization: unified AGN model



Key supporting evidence

If you look at the *polarized component* of the light emitted by Seyfert 2 galaxy, it is more similar to that of a Seyfert 1 galaxy

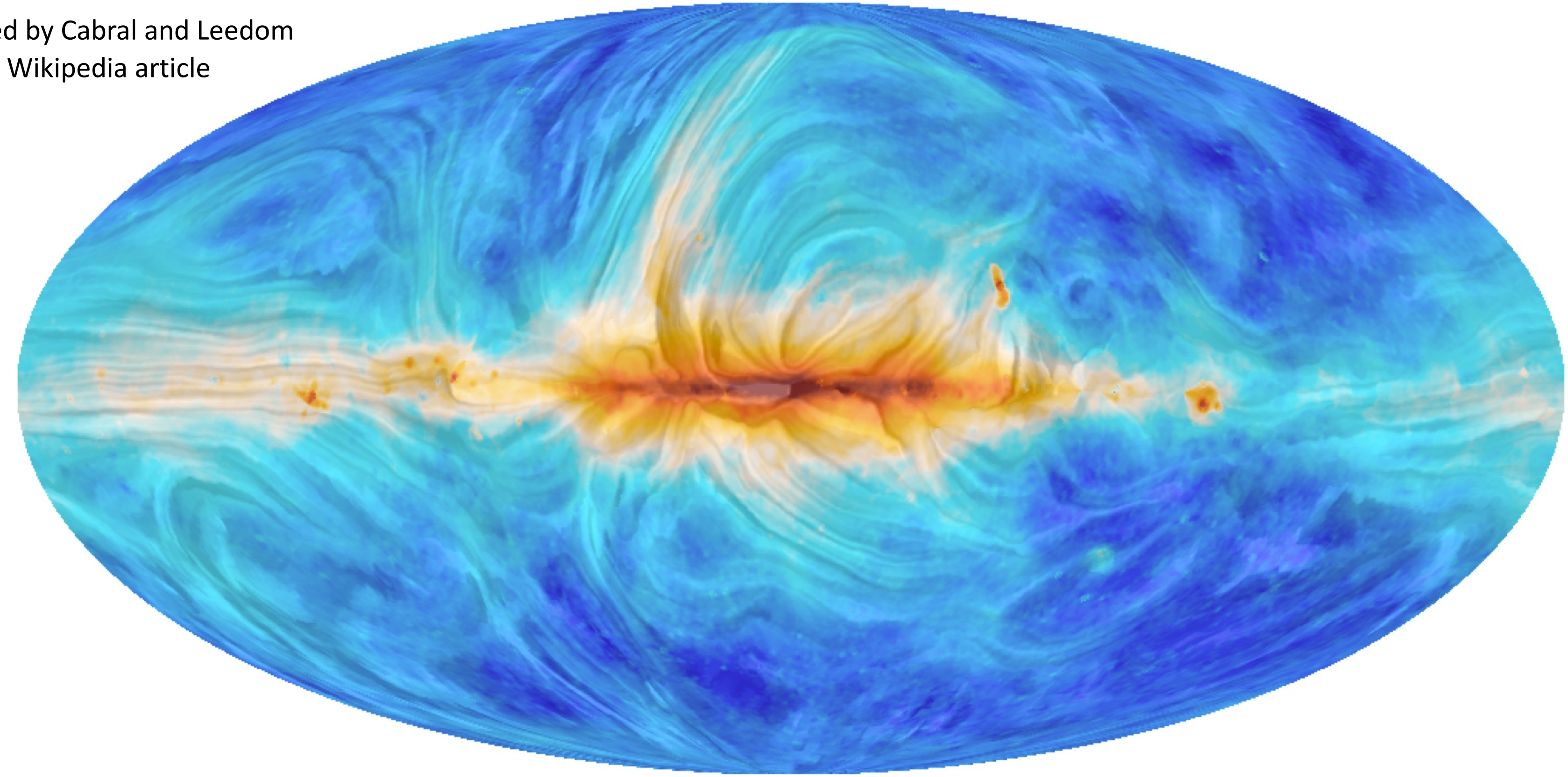
Polarization is imparted by scattering off gas and dust

(Opposite example: seeing fish in a lake when wearing polarizing sunglasses)

Polarization as a probe of magnetic fields, which show large scale Galactic structure

Swirls show B-field direction (Line integral convolution map*) 30 GHz synchrotron radiation

*Invented by Cabral and Leedom
Look at Wikipedia article



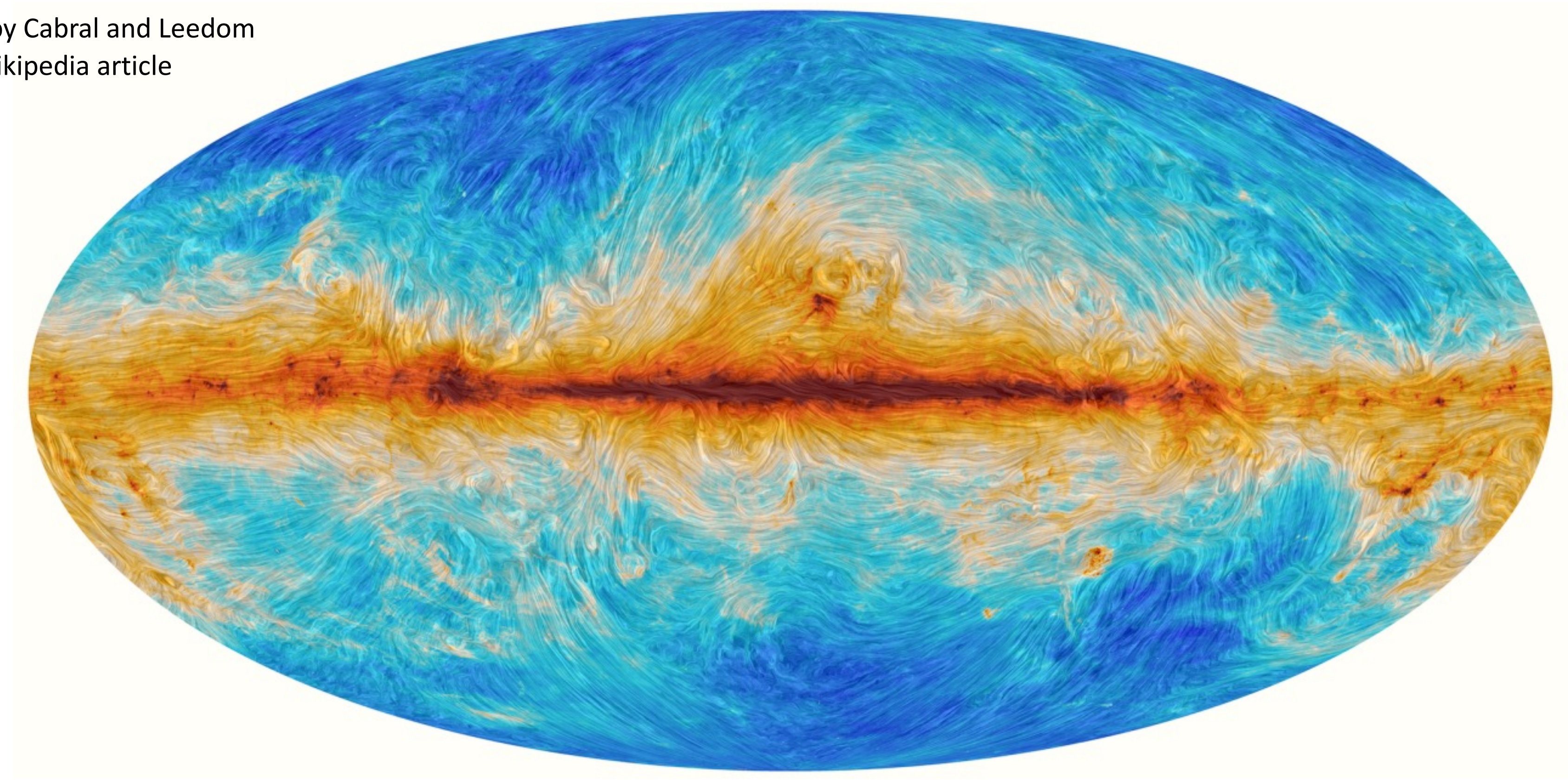
Polarized emission from cosmic-rays in our Galaxy as measured by *Planck* at 30 GHz

Charged particles orbit magnetic field lines and emit radiation with with its E-field perpendicular to the interstellar B-field

Polarization as a probe of magnetic fields, which show large scale Galactic structure

Swirls show B-field direction (Line integral convolution map*) 353 GHz dust emission

*Invented by Cabral and Leedom
Look at Wikipedia article

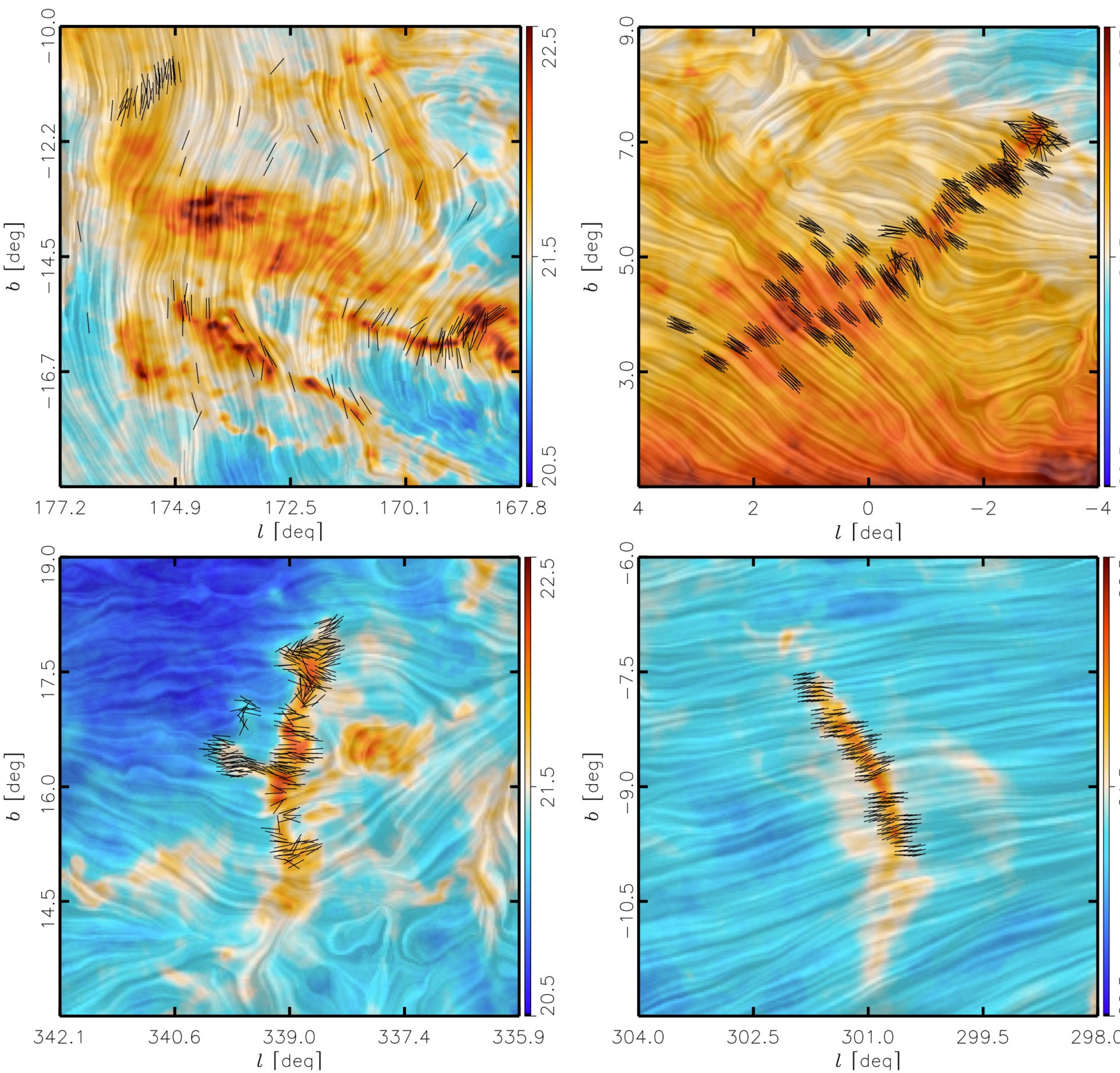


Polarized emission from dust grains in our Galaxy as measured by *Planck* at 353 GHz

Grains are elongated and preferentially aligned perpendicular to the interstellar B-field

➔ they emit thermal radiation with its E-field perpendicular to the IS B-field

…. and as probe of magnetic fields in individual interstellar gas clouds



Smaller scale maps from Soler+ 2016

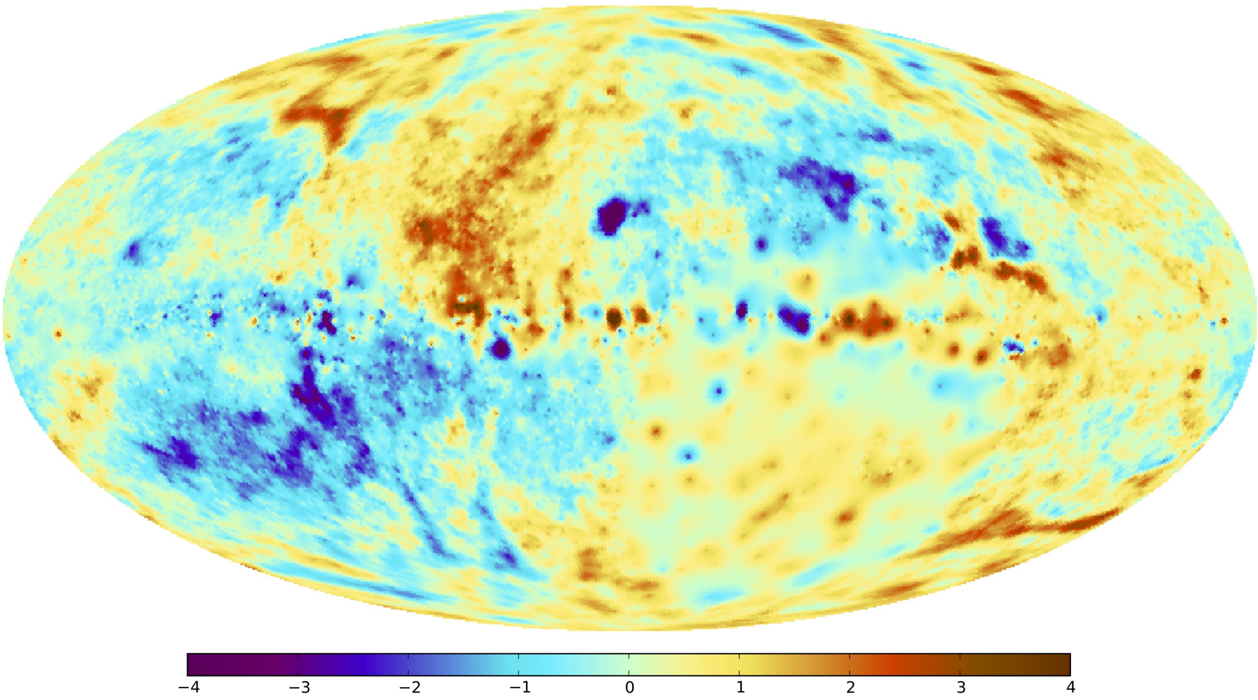
Black lines show the directions of the B-field inferred from the IR polarization of background stars.

In this case, IR radiation with the electric field *perpendicular* to the interstellar B-field direction is preferentially absorbed

Faraday rotation

B-fields *along* our line of sight causes "Faraday rotation"

All-sky map from Oppermann+ 2012



As polarized radiation propagates through a magnetized plasma, the polarization direction is rotated through an angle proportional to $\lambda^2 \int B_\parallel n_e \, ds$

Lecture 9
Polarization

Goals: understand

Stokes parameters

READING: *R&L* 2.4

Polarization of a monochromatic wave

The electric field is written $\boldsymbol{E} = \boldsymbol{E_0} e^{i(\boldsymbol{k}.\boldsymbol{x}-kct)}$

with the understanding that this really means

$\boldsymbol{E} = \mathrm{Re}\{\boldsymbol{E_0} e^{i(\boldsymbol{k}.\boldsymbol{x}-kct)}\}$

$\boldsymbol{E_0}$ is a complex vector, which is perpendicular to the propagation direction, $\widehat{\boldsymbol{k}}$. Let's orient the *z*-axis along the propagation direction, so $\boldsymbol{E_0}$ is in the xy-plane

$\boldsymbol{E_0} = \widehat{\boldsymbol{x}} E_{0x} + \widehat{\boldsymbol{y}} E_{0y}$, where E_{0x} and E_{0y} are complex

Note we never need to treat the B field separately, since

$\boldsymbol{B} = \boldsymbol{E} \times \widehat{\boldsymbol{k}}$ has the same magnitude and phase

Polarization of a monochromatic wave

We can write the complex numbers E_{0x} and E_{0y} as follows

$E_{0x} = \mathcal{E}_x e^{i\phi_x}$ $\quad E_{0y} = \mathcal{E}_y e^{i\phi_y}$ with $\mathcal{E}_x, \mathcal{E}_y, \phi_x, \phi_y \in \Re$

At *z = 0*, we then get

$$\boldsymbol{E} = \hat{\boldsymbol{x}}\, \mathcal{E}_x \cos(\omega t - \phi_x) + \hat{\boldsymbol{y}}\, \mathcal{E}_y \cos(\omega t - \phi_y)$$

Polarization of a monochromatic plane wave

Case 1: E_{0x}/E_{0y} is real

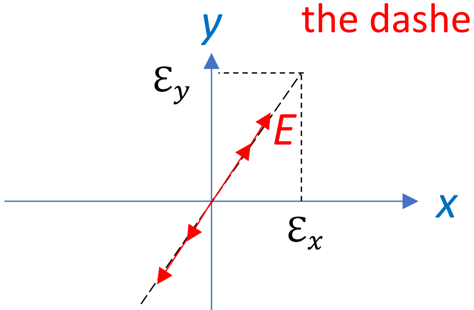
➔ $\phi_x = \phi_y = \phi$

➔ The *x* and *y* components of ***E*** vary in phase

➔ $\boldsymbol{E} = (\hat{\boldsymbol{x}}\, \mathcal{E}_x + \hat{\boldsymbol{y}}\, \mathcal{E}_y)\cos(\omega t - \phi)$

➔ Linearly polarized radiation

SHM along the dashed line



Case 2: E_{0x}/E_{0y} is $\pm i$

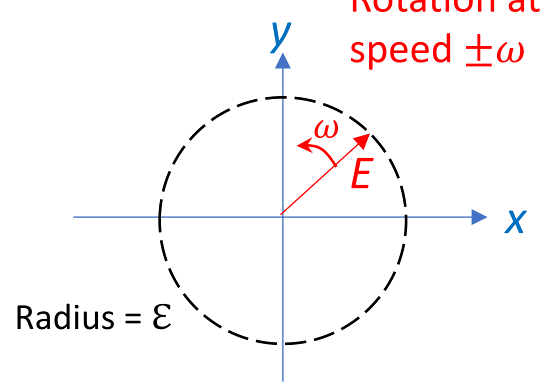
➔ $\phi_x = \phi_y \pm \frac{\pi}{2}$ and $\mathcal{E}_x = \mathcal{E}_y = \mathcal{E}$

➔ The *x* and *y* components are 90^0 out of phase

➔ $E = \mathcal{E}\hat{\boldsymbol{x}}\cos(\omega t - \phi_x) \pm \mathcal{E}\hat{\boldsymbol{y}}\sin(\omega t - \phi_x)$

➔ Circularly polarized radiation

Rotation at angular speed $\pm\omega$



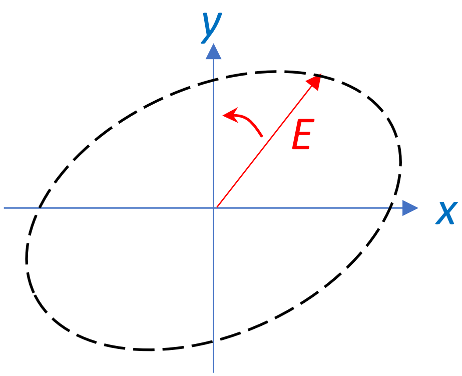
Polarization of a monochromatic plane wave

Case 3: E_{0x}/E_{0y} is complex (or imaginary but $\neq \pm i$)

➔ $\phi_x = \phi_y - \Delta\phi$

➔The E-field rotates around an ellipse

➔ Elliptically polarized radiation



Most generally, the polarization of a monochromatic wave is characterized by three parameters: $\Delta\phi$, $\mathcal{E}_x$, $\mathcal{E}_y$

This makes sense, because three parameters are needed to describe an ellipse: semi-major axis, axial ratio, and orientation

Stokes parameters

We define the Stokes parameters as follows

$$I \equiv E_{0x} E_{0x}^{\;*} + E_{0y} E_{0y}^{\;*} \qquad = \mathcal{E}_x^{\;2} + \mathcal{E}_y^{\;2}$$

$$Q \equiv E_{0x} E_{0x}^{\;*} - E_{0y} E_{0y}^{\;*} \qquad = \mathcal{E}_x^{\;2} - \mathcal{E}_y^{\;2}$$

$$U \equiv E_{0x} E_{0y}^{\;*} + E_{0y} E_{0x}^{\;*} \qquad = 2\mathcal{E}_x\mathcal{E}_y \cos\Delta\phi$$

$$V \equiv i(E_{0y} E_{0x}^{\;*} + E_{0x} E_{0y}^{\;*}) = 2\mathcal{E}_x\mathcal{E}_y \sin\Delta\phi$$

There are 4 parameters, but only 3 are needed to define an ellipse

So for this case of monochromatic radiation, there is a redundant information ➔ there must be a relationship between them, and indeed

$$Q^2 + U^2 + V^2 = (\mathcal{E}_x^{\;2} - \mathcal{E}_y^{\;2})^2 + 4\mathcal{E}_x^{\;2}\mathcal{E}_y^{\;2}\cos^2\Delta\phi + 4\mathcal{E}_x^{\;2}\mathcal{E}_y^{\;2}\sin^2\Delta\phi$$

$$= (\mathcal{E}_x^{\;2} + \mathcal{E}_y^{\;2})^2 = I^2$$

As we'll see, this relationship need not apply when we superpose waves at slightly different frequencies or take a time average when the Stokes parameters are varying

Stokes parameters: meaning

$$I \equiv E_{0x} E_{0x}^{\ \ *} + E_{0y} E_{0y}^{\ \ *} = \mathcal{E}_x^{\ 2} + \mathcal{E}_y^{\ 2}$$

$$Q \equiv E_{0x} E_{0x}^{\ \ *} - E_{0y} E_{0y}^{\ \ *} = \mathcal{E}_x^{\ 2} - \mathcal{E}_y^{\ 2}$$

$$U \equiv E_{0x} E_{0y}^{\ \ *} + E_{0y} E_{0x}^{\ \ *} = 2\mathcal{E}_x\mathcal{E}_y \cos\Delta\phi$$

$$V \equiv i(E_{0y} E_{0x}^{\ \ *} + E_{0x} E_{0y}^{\ \ *}) = 2\mathcal{E}_x\mathcal{E}_y \sin\Delta\phi$$

$I = \frac{8\pi}{c}\langle S\rangle$ is proportional to the total flux

$V \propto \sin\Delta\phi$ is called the “circularity parameter”

$V = 0$ for linear polarized radiation

$V = \pm I$ for circularly polarized radiation

V/I determines the axial ratio

U and Q determine the orientation of the ellipse

Partially-polarized radiation

So far, we have considered radiation in which the polarization state is unchanging and the Stokes parameters are constant.

For simplicity, let's assume the flux is constant but the polarization state changes on some timescale Δt that is much smaller than the observation period, *T*, but larger than the period of oscillation, $2\pi/\omega$.

We will observe time averaged values of the Stokes parameters,

$\langle I \rangle_T$, $\langle Q \rangle_T$, $\langle U \rangle_T$, and $\langle V \rangle_T$

For constant I, ${\langle I \rangle_T}^2 = \langle I^2 \rangle_T$

But for the parameters that vary, we have

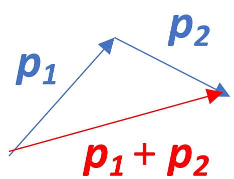
${\langle U \rangle_T}^2 \leq \langle U^2 \rangle_T$, ${\langle Q \rangle_T}^2 \leq \langle Q^2 \rangle_T$, ${\langle V \rangle_T}^2 \leq \langle V^2 \rangle_T$

Hence, ${\langle I \rangle_T}^2 \geq {\langle U \rangle_T}^2 + {\langle Q \rangle_T}^2 + {\langle V \rangle_T}^2$

Partially-polarized radiation

We can also think of this graphically by drawing 3D-vectors to represent $(U, Q,, V)$

So long as $U, Q,$ and V are constant, $I = \sqrt{U^2 + Q^2 + V^2}$ is the length of such a vector



Suppose we have two such vectors, $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$ representing the Stokes parameters during two equal time periods.

The time-averaged I is ½($|\boldsymbol{p_1}| + |\boldsymbol{p_2}|$)

The time averaged (U,Q,V) is simply ½$(\boldsymbol{p_1} + \boldsymbol{p_2})$, and thus the time-averaged quantity $\sqrt{\langle Q\rangle^2 + \langle U\rangle^2 + \langle V\rangle^2}$ *is* ½$|\boldsymbol{p_1} + \boldsymbol{p_2}|$

The triangle inequality, $|\boldsymbol{p_1} + \boldsymbol{p_2}| \leq |\boldsymbol{p_1}| + |\boldsymbol{p_2}|$, tells us $\sqrt{\langle Q\rangle^2 + \langle U\rangle^2 + \langle V\rangle^2} \leq \langle I\rangle$

The same argument applies to the superposition of two monochromatic waves of similar frequency but different polarization state

Fractional polarization, Π

We define the fractional polarization, Π, as follows

$$\Pi \equiv \frac{\sqrt{U^2 + Q^2 + V^2}}{I}$$

If the polarization state is constant, $\Pi = 1$

If the polarization state varies completely randomly, $\Pi = 0$

In general, Π can lie anywhere between 0 and 1

Lecture 10
Retarded and Lienard-Weichart Potentials

Goals: understand

The retarded potentials
The L-W potentials for a point charge
Potentials for a collection of charges
Potential for a collection of charges
The wave zone

READING: *R&L* 3.1, 3.2

Solution to Maxwell's equation with charges and currents

We wish to solve Maxwell's equations for non-zero ρ and $\boldsymbol{j}$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}\right)\begin{matrix}\phi \\ \boldsymbol{A}\end{matrix} = -4\pi \begin{matrix}\rho \\ \boldsymbol{j}/c\end{matrix}$$

Let's focus first on the equation for ϕ

To solve this *inhomogenous* equation, we first determine the Green's function i.e. the solution for the case of a delta function at location $\boldsymbol{x}'$

$$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t} = -4\pi Q(t)\delta(\boldsymbol{x} - \boldsymbol{x}')$$

The solution must be a spherical wave centered on $\boldsymbol{x}'$

$\phi(\boldsymbol{x}, t) = \frac{1}{R} f(t - R/c)$, where $R = |\boldsymbol{x} - \boldsymbol{x}'|$

Check: $\boldsymbol{\nabla}^2\phi = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left[-\frac{f}{R^2} - \frac{f\prime}{Rc}\right]\right) = \frac{1}{R^2}\left[\frac{f\prime}{c} - \frac{f'}{c} + \frac{Rf\prime\prime}{c^2}\right] = \frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t}$

Solution to Maxwell's equation with charges and currents

$$\phi(\boldsymbol{x}, t) = \frac{1}{R} f(t - R/c)$$

But how do we determine f?

Consider the limit of small R. Then the spatial derivatives dominate those with respect to time, and the equation becomes $\boldsymbol{\nabla}^2\phi = -4\pi Q(t)\delta(\boldsymbol{x} - \boldsymbol{x}')$

But we know the solution to that from electrostatics: $\phi = \frac{Q(t)}{R}$

$$\lim_{R\to 0}\left(\frac{1}{R} f(t - R/c)\right) = \frac{Q(t)}{R} \Longrightarrow f = Q$$

The exact solution for all R is therefore $\phi(\boldsymbol{x}, t) = \frac{1}{R} Q(t - R/c) = \frac{1}{R} Q(t_{ret})$

where the "retarded time" $t_{ret} \equiv t - R/c$

Solution to Maxwell's equation with charges and currents

Because Maxwells' equations are linear, this implies that the general solution is

$$\phi(\boldsymbol{x},t) = \int \frac{\rho(\boldsymbol{x}',t-|\boldsymbol{x}-\boldsymbol{x}'|/\boldsymbol{c})}{|\boldsymbol{x}-\boldsymbol{x}'|}dV'$$

$$\phi(\boldsymbol{x},t) = \int \frac{[\rho]}{|\boldsymbol{x}-\boldsymbol{x}'|}dV'$$

where [ ] indicates a retarded value (evaluated at the t_{ret} appropriate to each location in V')

An identical argument yields

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{c}\int \frac{[\boldsymbol{j}]}{|\boldsymbol{x}-\boldsymbol{x}'|}dV'$$

These are called the retarded potentials

Solution to Maxwell's equation with charges and currents

$$\phi(\boldsymbol{x},t) = \int \frac{[\rho]}{|\boldsymbol{x} - \boldsymbol{x}'|} dV'$$

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{c}\int \frac{[\boldsymbol{j}]}{|\boldsymbol{x} - \boldsymbol{x}'|} dV'$$

Interpretation: there is a time lag in the propagation of information about ρ and $\boldsymbol{j}$

The information only propagates at the speed of light

Note: the equations allow a solution where ρ and $\boldsymbol{j}$ are to be evaluated at $t + |\boldsymbol{x} - \boldsymbol{x}'|$/c , i.e at some *future* time. [The *outgoing* spherical wave solution we adopted could be replaced by an incoming wave $\frac{1}{R} f(t + R/c)$]

But this can be rejected on the grounds of *causality,* because it would be inconsistent with the notion that charges and currents *cause* electric and magnetic fields

The potential for a moving point charge

There's a clever trick for enforcing the retarded time within the integrals for the retarded potentials. We can write

$$\phi(\boldsymbol{x},t) = \int \frac{\rho(\boldsymbol{x}', t_{ret})}{|\boldsymbol{x}' - \boldsymbol{x}|} dV' = \int \frac{\rho(\boldsymbol{x}', t')}{|\boldsymbol{x}' - \boldsymbol{x}|} \delta(t' - t_{ret}) dV' dt'$$

We are now ready consider the potentials associated with a moving point charge q which is located at position $\boldsymbol{x}' = \boldsymbol{r}(t))$

For this charge, we have $\quad \rho(\boldsymbol{x}', t) = q\delta(\boldsymbol{x}' - \boldsymbol{r}(t))$

$$\boldsymbol{j}(\boldsymbol{x}', t) = q\boldsymbol{v}(t)\delta(\boldsymbol{x}' - \boldsymbol{r}(t))$$

where $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t)$

For the point charge, we then find

$$\phi(\boldsymbol{x},t) = \int \frac{q\delta(\boldsymbol{x}' - \boldsymbol{r}(t'))}{|\boldsymbol{x}' - \boldsymbol{x}|} \delta(t' - t_{ret}) dV' dt'$$

The potential for a moving point charge

We can first perform the integration dV' to obtain

$$\phi(\boldsymbol{x},t) = \int \frac{q\delta(\boldsymbol{x}' - \boldsymbol{r}(t'))}{|\boldsymbol{x}' - \boldsymbol{x}|}\delta(t' - t_{ret})dV'dt' = \int \frac{q}{|\boldsymbol{r}(t') - \boldsymbol{x}|}\delta(t' - t_{ret})dt'$$

$$= \frac{q}{|\boldsymbol{r}(t_{ret}) - \boldsymbol{x}|}\int \delta(t' - t + |\boldsymbol{r}(t') - \boldsymbol{x}|/c)dt'$$

$$= \frac{q}{R(t_{ret})}\int \delta(t' - t + R(t')/c)dt'$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{x}$ and $R = |\boldsymbol{R}|$

Subtle but very important point: this integral of a delta function is *not* unity:

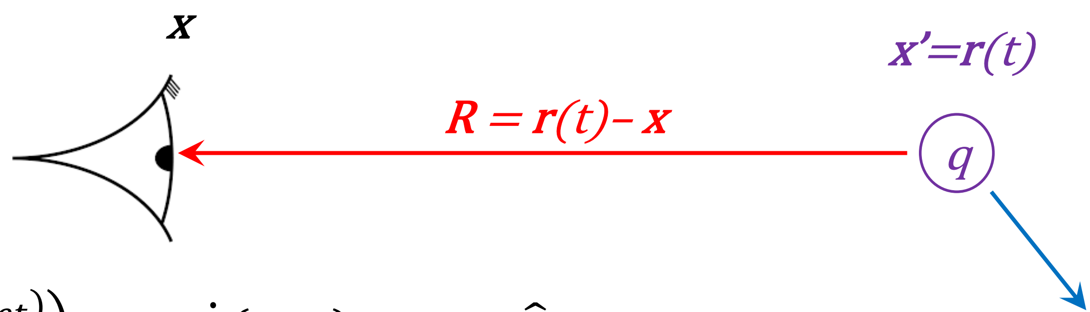
$$\int \delta(y)dx = \frac{\int \delta(x)dx}{\kappa} = \frac{1}{\kappa}$$

where

$$\kappa = \left(\frac{dy}{dx}\right)_{y=0} = \left(1 + \frac{\dot{R}(t')}{c}\right)_{t'=t_{ret}} = \left(1 + \frac{\dot{R}(t_{ret})}{c}\right)$$

The potential for a moving point charge

Picture:



Hence, $\kappa = \left(1 + \frac{\dot{R}(t_{ret})}{c}\right)$ with $\dot{R}(t_{ret}) = -v.\hat{R}$

$$\phi(\boldsymbol{x}, t) = \frac{q}{\kappa(t_{ret})R(t_{ret})} = \left[\frac{q}{\kappa R}\right]$$

And, by an identical argument

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{q\boldsymbol{v}}{\kappa(t_{ret})R(t_{ret})} = \left[\frac{q\boldsymbol{v}}{c\kappa R}\right]$$

with $\kappa = (1 - v.\hat{R}/c)$

These are called the Lienard-Weichart potentials

Notes on the Lienard Wiechert potentials

$$\phi(\boldsymbol{x},t) = \frac{q}{\kappa(t_{ret})R(t_{ret})} = \left[\frac{q}{\kappa R}\right] \qquad \boldsymbol{A}(\boldsymbol{x},t) = \frac{q}{\kappa(t_{ret})R(t_{ret})} = \left[\frac{q\boldsymbol{v}}{c\kappa R}\right]$$

with $\kappa = (1 - \boldsymbol{v}.\widehat{\boldsymbol{R}}/c)$

1) For a stationary charge, $\kappa = 1$ and $R(t_{ret}) = R$ is constant

We get the familiar electrostatic result: $\phi = q/R \qquad \boldsymbol{A} = \boldsymbol{0}$

2) For moving charges, there are two additional effects

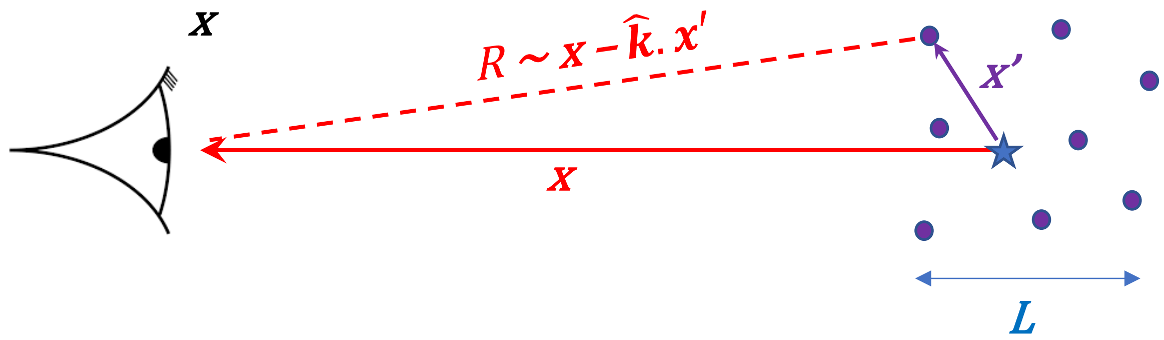
i) everything is evaluated at the retarded time

ii) the factor $\kappa \neq 1$ (unless the motion is perpendicular to $\hat{R}$)

This arises because $\int [\rho] dV' \neq \int \rho dV' = q$

3) The κ factor is extremely important for relativistic particles, where v is close to c and $1/\kappa$ becomes very large when $\boldsymbol{v}$ is in the direction $\widehat{\boldsymbol{R}}$ (i.e. moving towards us)

Potential due to collection of non-relativistic charges: the wave zone



Let's choose the origin ★ of our coordinate system within the charge collection of assumed size *L*. The observer is located at position $\boldsymbol{x}$, as always, and the charges are at location $\boldsymbol{x}'$ with $x' < L/2$

The observer is said to be in the "wave zone" if $x >> L$ and $x >> \lambda$, where λ is the characteristic wavelength of any radiation that these charges emit

In the "wave zone", we may make two approximations because $x >> L$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{c}\int \frac{\boldsymbol{j}(\boldsymbol{x}', t_{ret})}{|\boldsymbol{x} - \boldsymbol{x}'|} dV' \approx \frac{1}{c|\boldsymbol{x}|}\int \boldsymbol{j}(\boldsymbol{x}', t_{ret})\, dV'$$

$$t_{ret} = t - R/c \approx t - (x - \hat{\boldsymbol{k}}.x')$$

Here, $\hat{\boldsymbol{k}} = \hat{\boldsymbol{x}}$ is the unit vector pointing in the direction that waves would propagate from the source to the observer

Potential due to a collection of charges

Given the "wave zone" approximation, $\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{cx}\int \boldsymbol{j}(\boldsymbol{x}', t_{ret})\, dV'$

Let's consider the spatial derivatives of $\boldsymbol{A}$ at the observer's location

$$\frac{\partial A_i}{\partial x_j} = -\frac{1}{cx^2}\frac{\partial x}{\partial x_j}\int j_i\, dV' + \frac{1}{cx}\int \frac{\partial j_i}{\partial t}\frac{\partial t_{ret}}{\partial x_j}\, dV'$$

We can also determine that

$$\frac{\partial x}{\partial x_j} = \frac{\partial}{\partial x_j}\sqrt{x_k x_k} = \frac{x_j}{x} = \widehat{k}_j$$

and

$$\frac{\partial t_{ret}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(t - \frac{1}{c}\left(x - \widehat{\boldsymbol{k}}.\, x'\right)\right) = -\frac{1}{c}\frac{x_j}{x} = -\frac{\widehat{k}_j}{c}$$

so

$$\frac{\partial A_i}{\partial x_j} = -\left(\frac{A_i}{x} + \frac{\mathbf{1}}{c}\frac{\partial A_i}{\partial t}\right)\widehat{k}_j$$

Potential due to a collection of charges

To get this equation $\frac{\partial A_i}{\partial x_j} = -\left(\frac{A_i}{x} + \frac{\mathbf{1}}{c}\frac{\partial A_i}{\partial t}\right)\widehat{k_j}$ we have so far only assumed *x* >> *L*

Note the two terms have different dependences on our distance from the source

First term is $\propto \frac{A_i}{R} \propto \frac{1}{R^2}$ and $B \propto \frac{1}{R^2}$ ➔ $u \propto \frac{1}{R^4}$

This is the standard result from electro/magnetostatics

However, the second term is $\propto \frac{A_i}{R^0} \propto \frac{1}{R^1}$ and $B \propto \frac{1}{R^1}$ ➔ $u \propto \frac{1}{R^2}$

Electro*dynamics,* with time-varying A_i caused by time-varying currents (i.e. *accelerating* charges), can transport energy over large distances

In the wave zone, where x >> $2\pi\lambda$, the second term has a magnitude ≫ first term

$$\frac{1}{c}\frac{\partial A_i}{\partial t} \sim \frac{\omega}{c} A_i = \frac{1}{2\pi\lambda} A_i \gg \frac{A_i}{x}$$

Thus, we find $\frac{\partial A_i}{\partial x_j} = -\frac{\hat{k}_j}{c}\frac{\partial A_i}{\partial t}$. Similar reasoning ➔ $\frac{\partial \phi}{\partial x_j} = -\frac{\hat{k}_j}{c}\frac{\partial \phi}{\partial t}$.

Lecture 11

Wave zone, dipole approximation, Thomson scattering

Goals: understand

The dipole approximation

Larmor's formula

Thomson scattering

READING: *R&L* 3.3, 3.4

In the wave zone, the solution is a spherical transverse wave

$$\frac{\partial A_i}{\partial x_j} = -\frac{\hat{k}_j}{c}\frac{\partial A_i}{\partial t} \;\Rightarrow\; \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = -\frac{\hat{\boldsymbol{k}}}{c}\times\frac{\partial \boldsymbol{A}}{\partial t}$$

$$\frac{\partial \phi}{\partial x_j} = -\frac{\hat{k}_j}{c}\frac{\partial \phi}{\partial t} \;\Rightarrow\; \boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = \frac{1}{c}\frac{\partial \phi}{\partial t}\hat{\boldsymbol{k}} - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

But, $\frac{1}{c}\frac{\partial \phi}{\partial t} = -\boldsymbol{\nabla}.\boldsymbol{A} = \frac{\hat{\boldsymbol{k}}}{c}.\frac{\partial \boldsymbol{A}}{\partial t}$ (Lorentz Gauge)

Fancy way of writing 1

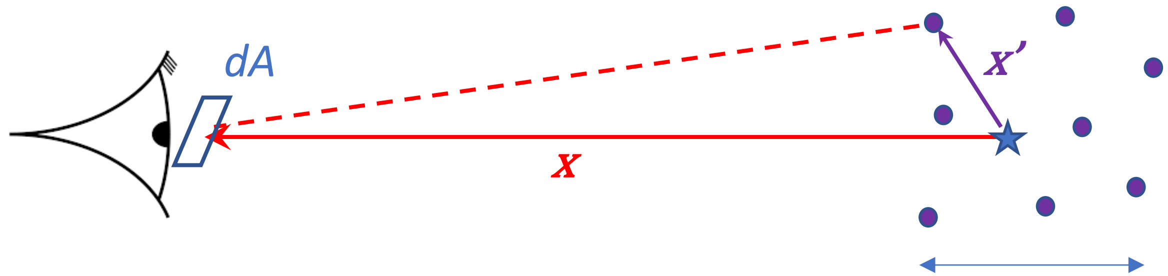
$$\boldsymbol{E} = \frac{1}{c}\frac{\partial \phi}{\partial t}\hat{\boldsymbol{k}} - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = \frac{\hat{\boldsymbol{k}}}{c}.\frac{\partial \boldsymbol{A}}{\partial t}\hat{\boldsymbol{k}} - (\hat{\boldsymbol{k}}.\hat{\boldsymbol{k}})\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = \left(\frac{\partial \boldsymbol{A}}{\partial t}\times\frac{\hat{\boldsymbol{k}}}{c}\right)\times\hat{\boldsymbol{k}} = \boldsymbol{B}\times\hat{\boldsymbol{k}}$$

Triple-product rule: $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a}.\boldsymbol{c}) - \boldsymbol{a}(\boldsymbol{b}.\boldsymbol{c})$

So, as before, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\hat{\boldsymbol{k}}$ are mutually perpendicular and $E = B$

The Poynting vector is $\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{c}{4\pi}B^2\,\hat{\boldsymbol{k}}$

Differential power



Let's consider an element of area, dA, at position $\boldsymbol{x}$

Energy passes through it at a rate $dP = S\,dA$

The solid angle subtended at the source is $d\Omega = dA/x^2$

So $\frac{dP}{d\Omega} = Sx^2 = \frac{c}{4\pi} B^2\, x^2 = \frac{c}{4\pi} \left|\frac{\partial \boldsymbol{A}}{\partial t} \times \frac{\widehat{\boldsymbol{k}}}{c}\right|^2 x^2 = \frac{\sin^2\Theta}{4\pi c} \left|\frac{\partial \boldsymbol{A}}{\partial t}\right|^2 x^2$

where Θ is the angle between $\frac{\partial \boldsymbol{A}}{\partial t}$ and $\widehat{\boldsymbol{k}}$

$\frac{dP}{d\Omega}$ is called the differential power (erg s^{-1} sr^{-1}) and depends on the direction $\widehat{\boldsymbol{k}}$

Differential power and the dipole approximation

Given our expression for the vector potential $A(\boldsymbol{x},t) = \frac{1}{c|\boldsymbol{x}|}\int \boldsymbol{j}(\boldsymbol{x}',t_{ret})\,dV'$

We find that $\frac{dP}{d\Omega} = \frac{\sin^2\Theta}{4\pi c^3}\left|\frac{\partial \boldsymbol{A}}{\partial t}\right|^2 = \frac{\sin^2\Theta}{4\pi c^3}\left|\frac{\partial}{\partial t}\int \boldsymbol{j}(\boldsymbol{x}',t_{ret})\,dV'\right|^2$

For the system to radiate, we clearly need *time-varying* currents

A considerably simplification occurs in the size of the emission region, *L*, is much *smaller* than the characteristic wavelength λ

In that limit, the "*dipole approximation*" is said to apply and phase differences across the source are negligible. Thus, we can assume the same retarded time for the entire emission region, and write

$$\frac{dP}{d\Omega} = \frac{\sin^2\Theta}{4\pi c^3}\left|\frac{\partial}{\partial t}\int \boldsymbol{j}(\boldsymbol{x}')\,dV'\right|^2$$

Dipole approximation

Suppose we have a collection of *N* charges, with individual charges q_i located at positions $\boldsymbol{x}_i$(t), where i ranges from 1 to *N*

In that case, $\boldsymbol{j}(\boldsymbol{x}') = \sum_1^N q_i\, \boldsymbol{v}_i\, \delta(\boldsymbol{x}_i - \boldsymbol{x}')$ where $\boldsymbol{v}_i = \dot{\boldsymbol{x}}_i$

Hence, $\int \boldsymbol{j}(\boldsymbol{x}')\, dV' = \sum_1^N q_i\, \dot{\boldsymbol{x}}_i$

and thus $$\frac{dP}{d\Omega} = \frac{\sin^2\Theta}{4\pi c^3}\left|\frac{\partial \boldsymbol{A}}{\partial t}\right|^2 = \frac{\sin^2\Theta}{4\pi c^3}\left|\frac{\partial}{\partial t}\int \boldsymbol{j}(\boldsymbol{x}')\, dV'\right|^2 = \frac{\sin^2\Theta}{4\pi c^3}\left|\sum_1^N q_i\, \ddot{\boldsymbol{x}}_i\right|^2$$

Defining the dipole moment $$\boldsymbol{d} = \sum_1^N q_i\, \boldsymbol{x}_i$$

We obtain $$\frac{dP}{d\Omega} = \frac{|\ddot{d}|^2 \sin^2\Theta}{4\pi c^3}$$ or $$\frac{dP}{d\Omega} = \frac{|\ddot{\boldsymbol{d}} \times \hat{\boldsymbol{k}}|^2}{4\pi c^3}$$

Dipole approximation

$$\frac{dP}{d\Omega} = \frac{|\ddot{d}|^2 \sin^2\Theta}{4\pi c^3} \quad \text{or} \quad \frac{dP}{d\Omega} = \frac{|\ddot{\boldsymbol{d}} \times \hat{\boldsymbol{k}}|^2}{4\pi c^3}$$

Integrating over solid angle, we obtain *Larmor's formula* for the total power

$$P = \int \frac{|\ddot{d}|^2 \sin^2\Theta}{4\pi c^3}\, d\Omega = \frac{|\ddot{d}|^2}{4\pi c^3} \int_{-1}^{1} 2\pi(1-\mu^2)\, d\mu = \frac{2|\ddot{d}|^2}{3c^3}$$

Comment about the various approximations

1) In the *wave zone approximation*, we assume $x >> L$ and $x >> \lambda$

but make no assumption about the relative magnitudes of L and λ

2) In the *dipole approximation,* we also assume $L << \lambda$

This is generally a good approximation for light atoms/molecules, which have typical size $L \sim a_0$ (Bohr radius) and electronic transitions with $\lambda \sim hc/\Delta E \sim hca_0/Ze^2$

Hence $L/\lambda \sim Ze^2/hc \sim Z/137$

Thomson scattering

We are now ready to consider the scattering of radiation by an electron (or other point charge)

We suppose a polarized EM wave is incident along the *z*-axis, causing the electron to move in simple harmonic motion along the *x*-axis

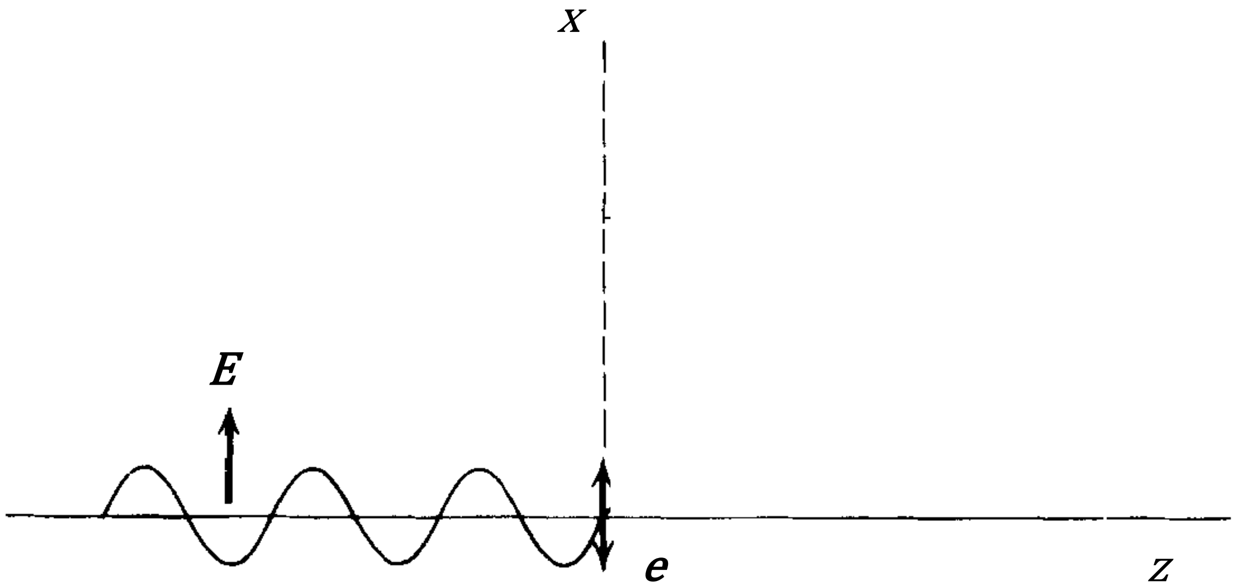


Figure 3.6 ***Scattering of polarized radiation by a charged particle.***

Thomson scattering: scattered power

If $\boldsymbol{E} = \hat{\boldsymbol{x}}\, \mathcal{E}_x \cos \omega t$

then $\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \sim \hat{\boldsymbol{x}}\, q\, \mathcal{E}_x \cos \omega t$

typically << 1

and $\ddot{\boldsymbol{x}} = \hat{\boldsymbol{x}}\, \frac{q}{m}\, \mathcal{E}_x \cos \omega t$

Hence $\ddot{\boldsymbol{d}} = q\ddot{\boldsymbol{x}} = \hat{\boldsymbol{x}}\, \frac{q^2}{m}\, \mathcal{E}_x \cos \omega t$

The power this electron radiates by virtue of its acceleration is

$$P = \frac{2|\ddot{d}|^2}{3c^3} = \frac{2q^4}{3m^2c^3}\, \mathcal{E}_x{}^2 \cos^2 \omega t = \frac{8\pi q^4}{3m^2c^4}\, \frac{c}{4\pi}\, \mathcal{E}_x{}^2 \cos^2 \omega t$$

incident flux, S

Thomson scattering: total cross-section

$$P = \frac{8\pi q^4}{3m^2c^4}\, S$$

The power radiated represents scattering, and thus the scattering cross-section for an electron is

$$\sigma_T = \frac{8\pi e^4}{3m e^2 c^4} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = \frac{8\pi {r_0}^2}{3} = 6.65 \times 10^{-25} \mathrm{cm}^2$$

σ_T is called the Thomson cross-section and r_0 = 2.82 x 10^{-13} cm is the "classical radius" of the electron

Protons also scatter radiation, but the cross-section is a factor $(\mathrm{m_p}/\mathrm{m_e})^2$ = 1837^2 smaller

Note that σ_T is independent of ν (although this classical treatment breaks down for $h\nu$ greater than $\sim m_e c^2$)

Thomson scattering: angular distribution

$\frac{dP}{d\Omega} = \frac{\left|\ddot{\boldsymbol{d}} \times \widehat{\boldsymbol{k}}\right|^2}{4\pi c^3}$, so the angular distribution has a $\sin^2\Theta$ dependence

Very important point: the relevant angle is with the *x*-axis (polarization direction) *not* the *z*-axis (incoming wave direction)

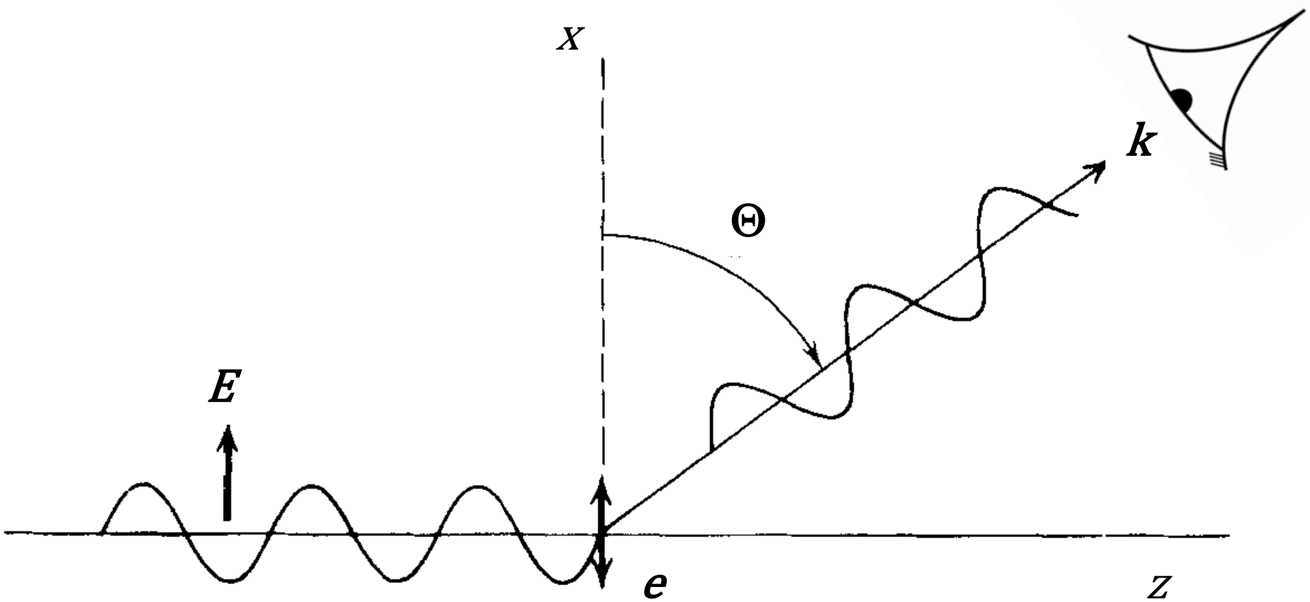


Figure 3.6 Scattering of polarized radiation by a charged particle.

Thomson scattering: differential cross-section

We may write the differential power of the scattered radiation

$$\frac{dP}{d\Omega} = \frac{|\ddot{\boldsymbol{d}}|^2}{4\pi c^3} \sin^2\Theta = \frac{e^4}{m_e{}^2 c^4} \frac{c}{4\pi} \mathcal{E}_x{}^2 \cos^2 \omega t \sin^2\Theta = \boldsymbol{S} r_0^2 \sin^2\Theta$$

terms of a differential cross-section

$$\frac{dP}{d\Omega} = \frac{d\sigma}{d\Omega} S$$

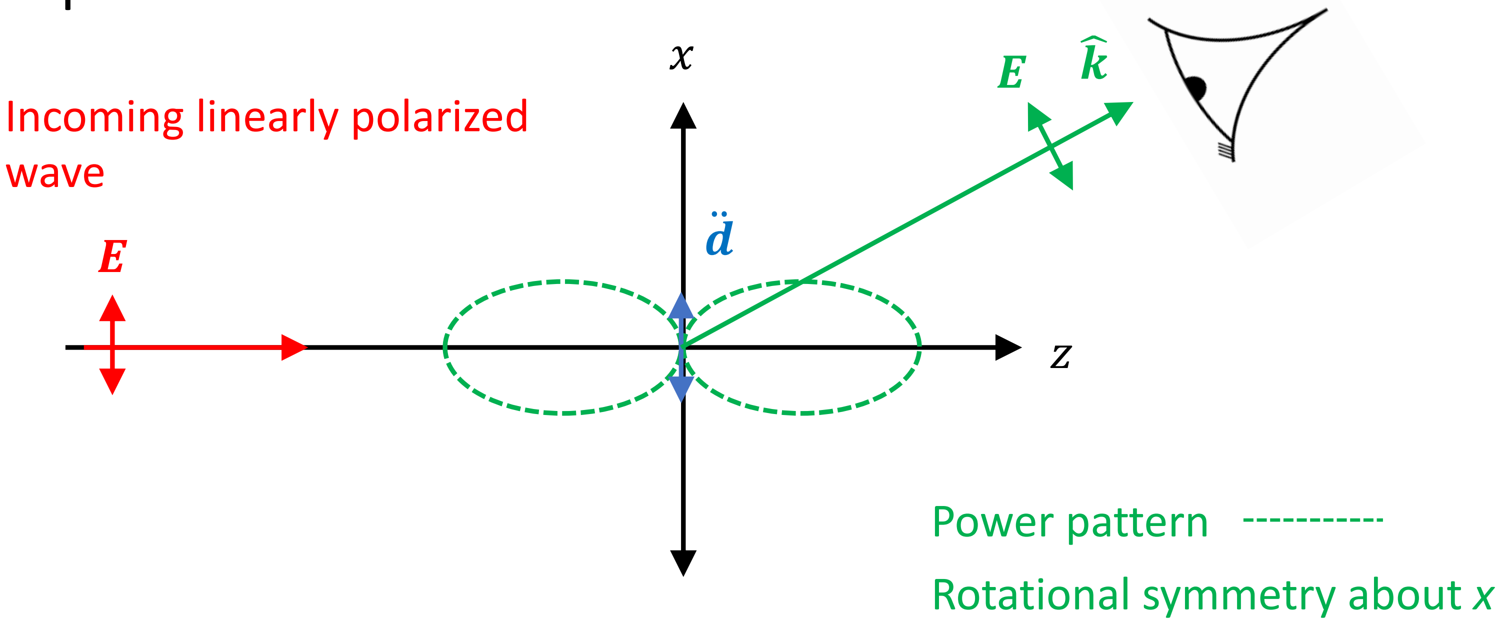
with $\frac{d\sigma}{d\Omega} = r_0^2 \sin^2\Theta$

Thomson scattering: angular distribution

Notes on the scattering of linearly-polarized radiation

1) The scattered radiation has a forward-backwards symmetry because $\sin^2\Theta = \sin^2(-\Theta)$

2) The scattered radiation is polarized with the ***E*** field in the *xz* plane

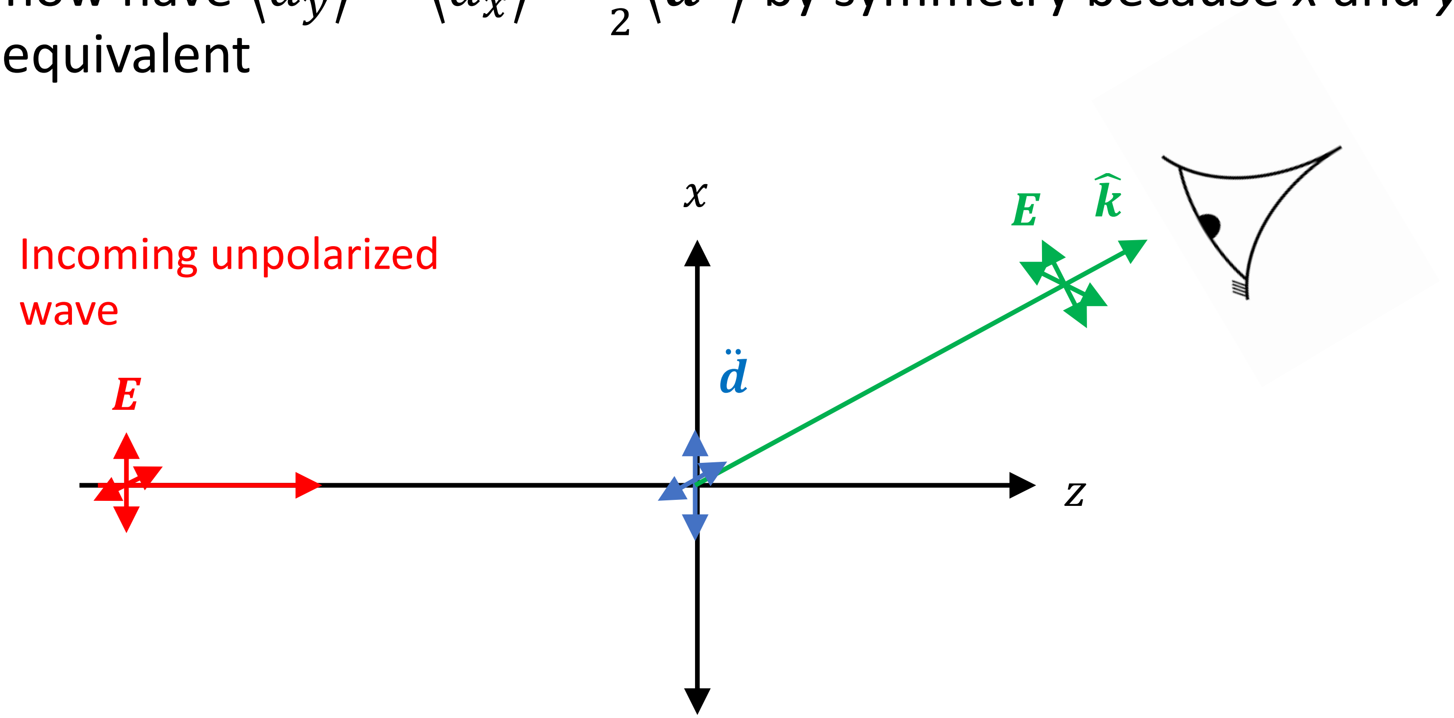


Thomson scattering: unpolarized radiation

So far, we have consider scattering of linearly-polarized radiation

With unpolarized radiation, the dipole moment has a *y*-component (in-and-out of the plane of the "paper")

We now have $\langle \ddot{d}_y^2 \rangle = \langle \ddot{d}_x^2 \rangle = \frac{1}{2}\langle \ddot{\boldsymbol{d}}^2 \rangle$ by symmetry because *x* and *y* are equivalent



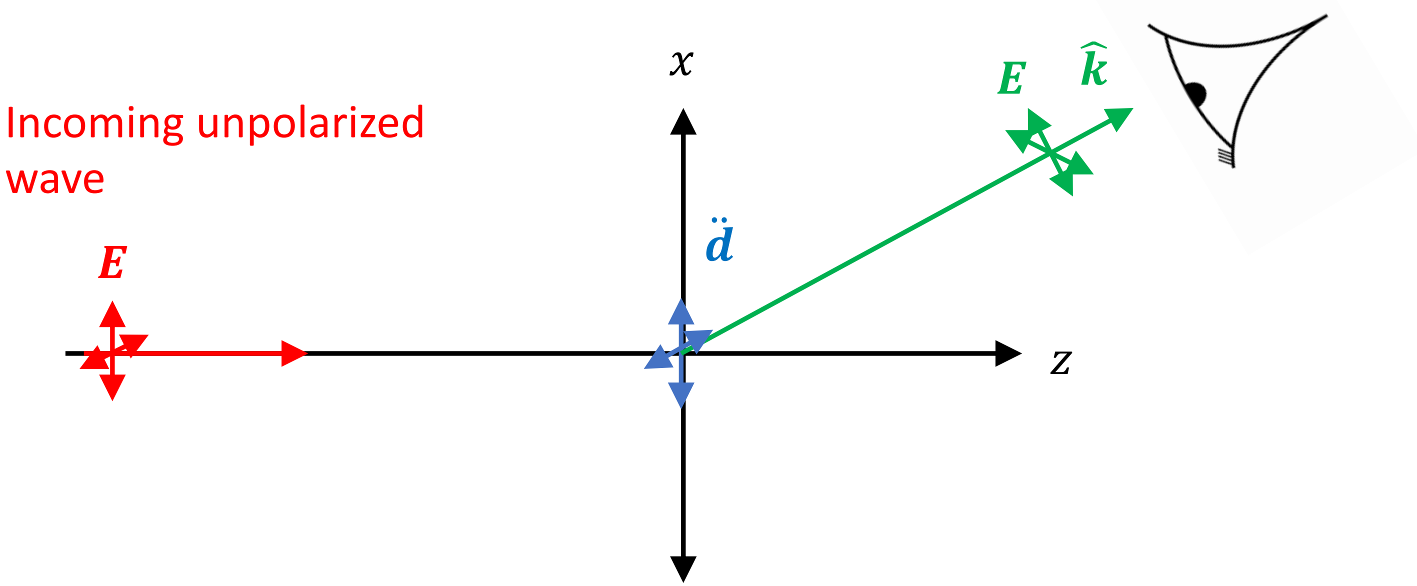
Thomson scattering: unpolarized radiation

The (time-averaged) differential power emitted is now

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\langle \ddot{d}_x^2 \rangle}{4\pi c^3} \sin^2\Theta + \frac{\langle \ddot{d}_y^2 \rangle}{4\pi c^3} \sin^2\frac{\pi}{2} = \frac{\langle \ddot{\boldsymbol{d}}^2 \rangle}{4\pi c^3} \left(\frac{1}{2}\sin^2\Theta + \frac{1}{2} \right)$$

and the differential scattering cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 (1 + \sin^2\Theta)$$



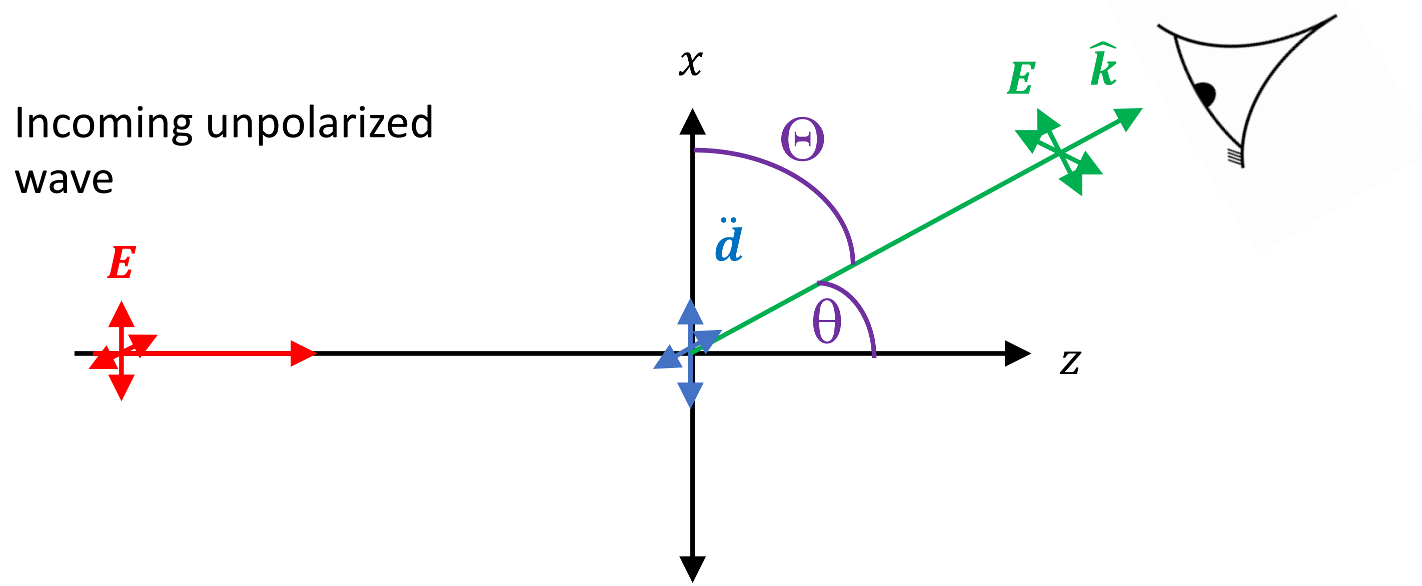
Thomson scattering: unpolarized radiation

Subtle point: the *x* axis is no longer a special axis. Any rotational symmetry has to exist about the *z*-axis.

The relevant angle is therefore θ, not Θ

(i.e. the angle to the *z* axis)

Thus, $\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 (1 + \cos^2\theta)$



Thomson scattering: unpolarized radiation

The total scattering cross-section is the same as for polarized radiation,

$$\sigma = \frac{1}{2} r_0^2 \int (1 + \cos^2\theta) d\Omega = \frac{1}{2} r_0^2 \int_{-1}^{1} 2\pi (1 + \mu^2)\, d\mu = \frac{8\pi r_0^2}{3} = \sigma_T$$

As we shall see, the scattered radiation can be partially-polarized, even when the incoming radiation is *unpolarized*

Lecture 12
Thomson scattering, charge in a harmonic potential

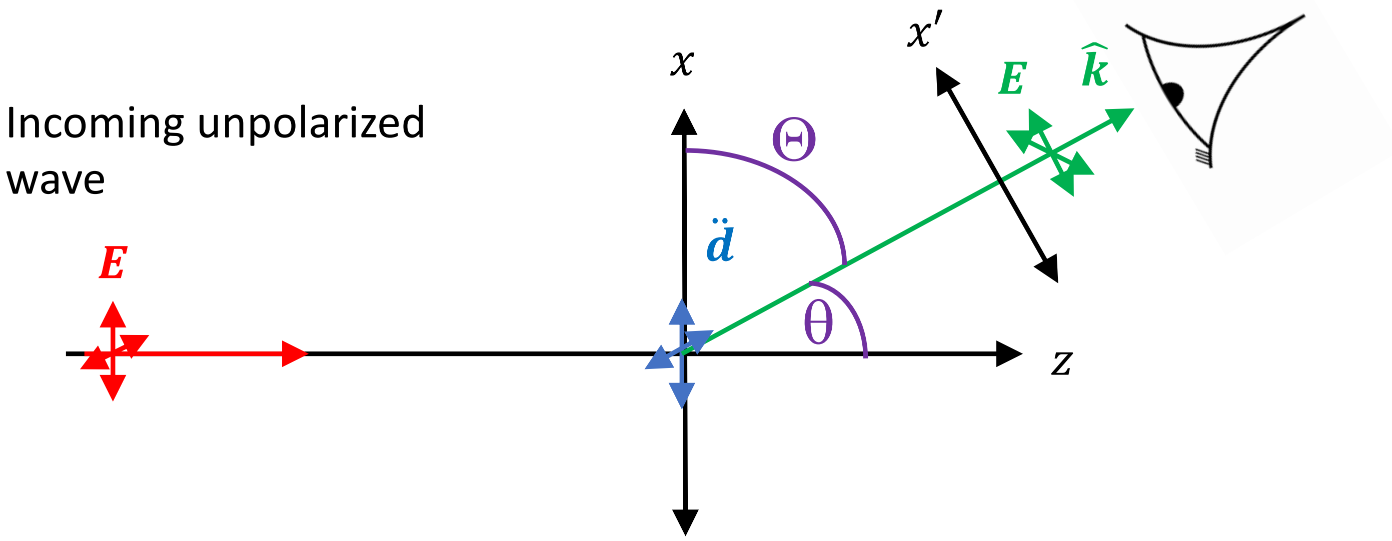
Goals: understand

Finish up Thomson scattering of unpolarized radiation

Scattering by a charge in a harmonic potential

Begin review of Special Relativity

Thomson scattering: unpolarized radiation



Define the x' axis as being rotated at angle θ, so it remains perpendicular to the *y* axis while also being perpendicular to $\hat{k}$

The time averaged value of ${\mathcal{E}_{x'}}^2$ is reduced by a factor $\cos^2\theta$ relative to the average value of ${\mathcal{E}_{y'}}^2$

➔ The *scattered radiation* has $\frac{I+Q}{I-Q} = \frac{{\mathcal{E}_{x'}}^2}{{\mathcal{E}_{y'}}^2} = \cos^2\theta$

➔ The degree of polarization $\Pi = \frac{Q}{I} = \frac{1-\cos^2\theta}{1+\cos^2\theta}$ is zero for $\theta = 0$ and 1 for $\theta = \pi/2$

Radiation from a harmonic oscillator

So far, we have considered the motion of a *free electron* upon which an EM wave is incident

Suppose the electron is bound in a harmonic potential (relevant to the case of atoms and molecules) and has a natural angular frequency of oscillation ω_0

Equation of motion: $m\ddot{x} + m\omega_0^2 x + \text{damping term} = qE_x$

The damping term is small but represents the energy loss due to radiation. We can usually treat this as a perturbation.

Let's consider first an undriven oscillator ($E_x = 0$) without damping

$m\ddot{x} + m\omega_0^2 x = 0 \quad \rightarrow x = x_0 e^{i\omega_0 t}$

where x_0 is the complex amplitude of the oscillator

Radiation from a harmonic oscillator

$x = x_0 e^{i\omega_0 t}$ ➔ dipole moment, $d = q x_0 e^{i\omega_0 t}$

And thus $\ddot{d} = -q\omega_0{}^2 x_0 e^{i\omega_0 t}$ (sinusoidal oscillation with amplitude $|x_0|$)

➔ $\langle \ddot{d}^2 \rangle = \frac{1}{2} q^2 \omega_0{}^4 |x_0|^2$

$\langle \cos^2 \omega_0 t \rangle$

Mean power radiated $\langle P \rangle = \frac{2\langle \ddot{d}^2 \rangle}{3c^3} = \frac{q^2 \omega_0{}^4 |x_0|^2}{3c^3}$

The particle energy, *E*, is the maximum value of $\frac{1}{2} m\dot{x}^2$, which is $\frac{1}{2} m\omega_0{}^2 |x_0|^2$

Hence, $\frac{dE}{dt} = -\frac{q^2 \omega_0{}^4 |x_0|^2}{3c^3} = -\frac{2\omega_0{}^2 q^2}{3mc^3} E = \frac{E}{\tau_{rad}}$

where the energy loss timescale $\tau_{rad} \equiv \frac{3mc^3}{2\omega_0{}^2 q^2} = \frac{3c}{r_0\, \omega_0{}^2}$

Radiation from a harmonic oscillator: energy loss

Key point:

$$\tau_{rad} \equiv -\frac{3mc^3}{2{\omega_0}^2 q^2} = \frac{3c}{2r_0\,{\omega_0}^2}$$

$\omega_0\tau_{rad} = \frac{3c}{2r_0\,\omega_0} = \frac{3\lambda_0}{4\pi r_0} \gg 1$ unless we are considering high-energy gamma-rays

➔ Fraction of energy lost per oscillation period is << 1

Can treat radiation as a perturbation

Energy decreases as $e^{-t/\tau_{rad}}$, so amplitude decreases as $e^{-t/(2\tau_{rad})}$

In other words, $x = x_0 e^{-\Gamma t/2}\, e^{i\omega_0 t}$

where the "damping constant," $\Gamma = 1/\tau_{rad}$

Equation of motion for a damped harmonic oscillator

As you probably remember from a sophomore course on waves,

$x = x_0 e^{-\Gamma t/2}\, e^{i\omega_0 t}$ is the solution to the equation of motion for the (undriven) damped harmonic oscillator

$$m(\ddot{x} + \Gamma\dot{x} + \omega_0^2 x) = 0$$

Note: It is shown in R&L 3.4 that the effect of radiation is to yield a reaction force that is proportional to $\dddot{x}$, the third time derivative of position.

This in only true in an average sense anyway, and since the damping term is a small perturbation, we have $\dddot{x} = -\omega_0^2 \dot{x}$

So $m(\ddot{x} + \Gamma\dot{x} + \omega_0^2 x) = 0$ is a very good approximation that is easy to work with

Equation of motion for a damped harmonic oscillator

We may now compute what happens when an EM wave is incident on the bound electron. We just add a "driving term" on the right-hand-side

$$(\ddot{x} + \Gamma\dot{x} + \omega_0^2 x) = \frac{F}{m} = -\frac{e}{m}E_{ox}\, e^{-i\omega t}$$

Here ω is the angular frequency of the incident wave, which (in general) differs from the natural frequency of oscillation ω_0

The solution is $x = x_0 e^{-i\omega t}$ + *(the decaying solution we obtained before)*

Here, E_{ox} and x_0 are complex as before, and the actual E field and displacement are given by the real parts of $E_{ox}\, e^{-i\omega t}$ and $x_0 e^{-i\omega t}$

i.e. $E_x = |E_{ox}| \cos{(\omega t - \delta_E)}$ and $x = |x_o| \cos{(\omega t - \delta_x)}$

Equation of motion for a damped harmonic oscillator

Substituting $x = x_0 e^{-i\omega t}$ into the equation of motion, we get

$$(-\omega^2 x_0 - i\Gamma\omega x_0 + \omega_0^2 x_0)e^{-i\omega t} = -\frac{eE_{ox}}{m}\, e^{-i\omega t}$$

$$\Rightarrow x_0 = \left(\frac{eE_{ox}}{m}\right)\frac{1}{(\omega_0^2 - \omega^2) - i\Gamma\omega}$$

$$\Rightarrow |x_0|^2 = \left(\frac{e^2|E_{ox}|^2}{m^2}\right)\frac{1}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2}$$

and

$$\delta_x = \delta_E - \tan^{-1}\left(\frac{\Gamma\omega}{\omega^2 - \omega_0^2}\right)$$

In resonance $(\omega = \omega_0)$, there's a $\frac{\pi}{2}$ phase difference between force and position

(➔ force in phase with *velocity*, maximizing the energy input to the system)

Scattering cross-section for a damped harmonic oscillator

We can now use the Larmor formula, as before, to determine the frequency dependence of the cross-section. The time-averaged power radiated is

$$\langle P\rangle = \frac{2\langle \ddot{d}\rangle^2}{3c^3} = \frac{2\langle e\ddot{x}\rangle^2}{3c^3} = \frac{e^2|x_0|^2\omega^4}{3c^3} \quad \text{using } \ddot{x} = -\omega^2\, x_0\, e^{-i\omega t}$$

Substituting for $|x_0|^2$ from the previous slide, we find

$$\langle P\rangle = \frac{e^2|x_0|^2\omega^4}{3c^3} = \left(\frac{e^4|E_{ox}|^2}{3c^3m^2}\right)\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2}$$

and dividing finally by $\langle S\rangle = \frac{c}{8\pi}\,|E_{ox}|^2$ we obtain the scattering cross-section

$$\sigma = \frac{\langle P\rangle}{\langle S\rangle} = \left(\frac{8\pi e^4}{3m^2c^2}\right)\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2} = \frac{\sigma_T\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2}$$

Frequency dependence

$$\sigma = \frac{\sigma_T \omega^4}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2} = \frac{\sigma_T \omega^4}{(\omega - \omega_0)^2(\omega + \omega_0)^2 + \Gamma^2\omega^2}$$

1) <u>High frequency limit</u> (ω >> ω_0): $\sigma \sim \sigma_T$

Same behavior as the unbound particle because the restoring force is negligible

2) <u>Low-frequency limit</u> (ω << ω_0): $\sigma \sim \frac{\omega^4}{\omega_0^4}\sigma_T \sim \frac{1}{\lambda^4}$

"Rayleigh scattering"

3) <u>Near resonance</u> ($\omega \sim \omega_0$): $\sigma \sim \frac{\sigma_T {\omega_0}^4}{4\omega_0^2\,(\omega - \omega_0)^2 + \Gamma^2 {\omega_0}^2}$

As you are asked to demonstrate in the next homework

$\sigma(\omega_0) = c_1\lambda_0^2$ and $\int \sigma(\nu)d\nu = c_2\left(\frac{e^2}{mc}\right)$ where c_1 and c_2 are constants (involving integers and π)

Review of SR: goals of the discussion

Our discussion of SR is motivated by the fact that relativistic charged particles are widespread in the Universe.

They are inevitably involved in two of the three emission processes we'll consider: synchrotron radiation and inverse Compton radiation.

So we'll need to understand

1) How the differential power $dP/d\Omega$ transforms as we go from one inertial reference frame to another

2) The dynamics of relativistic particles in a magnetic field

The fundamental postulates of SR

These are usually expressed as

1) The laws of physics are the same in any inertial reference frame (IRF)
2) The speed of light is the same in any inertial reference frame

The fundamental *object* in SR is the "event," which occurs at a particular spatial position (x, y, z) *and* at a particular time, t.

If we take two events, the emission and reception of a radio signal, the second postulate implies that

$$c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) = c^2\Delta t'^2 - (\Delta x'^2 + \Delta y'^2 + \Delta z'^2)$$

where (x, y, z, t) are the values measured in one IRF (call it *S*) and

and (x', y', z', t') are those measured in another (*S'*)

Four-vectors

Apart from the – sign, the "invariant" quantity $c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)$ looks a lot like the square of the length of a 4-D vector

The location of an event in *spacetime* may be expressed in two ways using the 4-vectors

$$x^\mu = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \quad \text{or} \quad x_\mu = \begin{bmatrix} -ct \\ x \\ y \\ z \end{bmatrix}$$

Here, μ can take 4 values, conventionally 0, 1, 2, 3

and $c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)$ may be written $\Delta x_\mu \Delta x^\mu$ using the summation convention. To obey Lorentz invariance, we only ever sum over one subscripted index and one superscripted

Lecture 13
Special relativity

Goals: review SR

4-vectors

Aberration and Doppler shift

Differential power received from a relativistic source

Relativistic beaming

Four-vectors

$$x^{\mu} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \qquad \text{or} \qquad x_{\mu} = \begin{bmatrix} -ct \\ x \\ y \\ z \end{bmatrix}$$

In this notation, a superscripted Greek letter index indicates a *contravariant* 4-vector, the meaning of which will be explained later, while a subscripted index indicates a *covariant* 4-vector

The contravariant and covariant forms differ only in the sign of the *0th* element

$$x_{\mu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \eta_{\mu\nu}\, x^{\nu}$$

("lowering the index")

where $\eta_{\mu\nu}$ is the Minkowski metric

The inverse transformation is $x^{\nu} = \eta^{\mu\nu}\, x_{\mu}$ ("raising the index")

Invariance: 3-vectors

We are familiar with the invariant properties of 3-vectors when we rotate the coordinate system

Angles and lengths are preserved under rotations, and therefore dot products are invariant

If $\boldsymbol{a}' = \boldsymbol{R}\,\boldsymbol{a} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$ (and $\boldsymbol{b}' = \boldsymbol{R}\,\boldsymbol{b}$)

(rotation about z axis through angle θ)

then $\boldsymbol{a}'.\boldsymbol{b}' = \boldsymbol{a}'_x\,\boldsymbol{b}'_x + \boldsymbol{a}'_y\,\boldsymbol{b}'_y + \boldsymbol{a}'_z\,\boldsymbol{b}'_z = \boldsymbol{a}_x\,\boldsymbol{b}_x + \boldsymbol{a}_y\,\boldsymbol{b}_y + \boldsymbol{a}_z\,\boldsymbol{b}_z = \boldsymbol{a}.\boldsymbol{b}$

and $|\boldsymbol{a}'| = \sqrt{\boldsymbol{a}'.\boldsymbol{a}'} = \sqrt{\boldsymbol{a}\,.\,\boldsymbol{a}} = |\boldsymbol{a}|$

Invariance: 3-vectors

The invariance of $\boldsymbol{a}.\boldsymbol{b}$ under rotation is linked to a mathematical property of $\boldsymbol{R}$. In the summation convention

$\boldsymbol{a}'.\boldsymbol{b}' = a'_i\, b'_i = R_{ij}R_{ik}a_j b_k$

The right hand side is equal to $a_j b_j$ because $R_{ij}\, R_{ik} = \delta_{jk}$

where δ_{jk} is the "Kronecker delta" (equals 1 when $j = k$ and 0 otherwise)

In matrix notation, this is saying $\boldsymbol{R}\boldsymbol{R}^{\boldsymbol{T}} = \boldsymbol{I}$

(R is said to be orthogonal: the transpose of R is equal to the inverse)

Invariance: 4-vectors

In SR, the analog of rotation is a "velocity boost" from one frame to another. The rotation matrix, ***R***, is replaced by the Lorentz transformation, so $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

with $\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ representing a velocity boost βc along the *x*-axis

and where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

Now, $\Delta x'^{\mu} \Delta x'_{\mu} = \Lambda^{\mu}{}_{\nu}\, \Delta x^{\nu} \eta_{\mu\rho} \Lambda^{\rho}{}_{\tau}\, \eta^{\tau\sigma} \Delta x_{\sigma} = \Delta x_{\mu}\, \Delta x^{\mu}$

because $\Lambda^{\mu}{}_{\nu}\, \eta_{\mu\rho}\, \Lambda^{\rho}{}_{\tau} = \eta_{\tau\rho}$ or equivalently $\boldsymbol{\Lambda}^{\boldsymbol{T}} \boldsymbol{\eta}\, \boldsymbol{\Lambda} = \boldsymbol{\eta}$

4-vectors

More generally, a 4-vector is any vector that transforms in accord with the Lorentz transformation, $V'^{\mu} = \Lambda^{\mu}{}_{v}\, V^{v}$

And the scalar product of any two 4-vectors, $V^{\mu}\, W_{\mu}$, is a Lorentz-invariant scalar. The postulates of SR then imply that the equations of physics can be written as 4-vector equations

Example: we may define the 4-velocity $U^{\mu} \equiv \dfrac{dx^{\mu}}{d\tau}$

where $d\tau = \sqrt{-dx^{\mu}dx_{\mu}/c^2}$ is the element of *proper time*

$d\tau$ is a Lorentz invariant scalar, so U^{μ} must clearly transform the way x^{μ} does (i.e. as a four-vector)

4-velocity

How does the 4-velocity relate to the three-velocity $\boldsymbol{u}$?

Well, $d\tau^2 = -dx^\mu dx_\mu / c^2 = dt^2 - (dx^2 + dy^2 + dz^2)/c^2$
$= dt^2(1 - u^2/c^2) = dt^2/\gamma^2 \Rightarrow d\tau = dt/\gamma$

Hence, $U^\mu = \frac{d}{d\tau}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \gamma \frac{d}{dt}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \gamma \begin{bmatrix} c \\ dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = \begin{bmatrix} \gamma c \\ \gamma \boldsymbol{u} \end{bmatrix}$

The product $U^\mu U_\mu$ should be Lorentz invariant and indeed it is

$U^\mu U_\mu = -\gamma^2 c^2 + \gamma^2 u^2 = -c^2$

Other physical quantities that are manifestly 4-vectors

4-acceleration: $a^{\mu} \equiv \dfrac{dU^{\mu}}{d\tau}$

Key feature of the 4–acceleration: $a^{\mu}U_{\mu} = \dfrac{dU^{\mu}}{d\tau}U_{\mu} = \dfrac{1}{2}\dfrac{d(U^{\mu}U_{\mu})}{d\tau} = 0$

a^{μ} and U_{μ} are orthogonal

In the instantaneous rest frame of the particle $a'^{\mu} = \begin{bmatrix} 0 \\ \boldsymbol{a}' \end{bmatrix}$

4-momentum: $p^{\mu} \equiv m_0 U^{\mu} = \begin{bmatrix} \gamma m_0 c \\ \gamma m_0 \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} m\, c \\ m\, \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} E/c \\ \boldsymbol{p} \end{bmatrix}$

where m_0 is the rest mass and $m = \gamma m_0$ is the "relativistic mass"

$-c^2 p^{\mu} p_{\mu} = E^2 - p^2 \mathrm{c}^2$ is Lorentz invariant and equal to $m_0{}^2 c^4$

4-momentum for photons

For a photon, U^{μ} is infinite and m_0 is zero

But the energy is $E = hv = \hbar\omega$ and the
3-momentum is $\boldsymbol{p} = \left(\frac{hv}{\mathrm{c}}\right)\widehat{\boldsymbol{k}} = \hbar\boldsymbol{k}$

Hence the 4-momentum is $p^{\mu} = \begin{bmatrix} \hbar\omega/c \\ \hbar\boldsymbol{k} \end{bmatrix}$

We may define the 4-wave-vector $k^{\mu} = \begin{bmatrix} \omega/c \\ \boldsymbol{k} \end{bmatrix}$ such that $p^{\mu} = \hbar k^{\mu}$

k^{μ} is *a "null vector"* with $k^{\mu}k_{\mu} = k^2 - \frac{\omega^2}{c^2} = 0$
In Minkowski space, we can have a non-zero vector with zero length

Invariance of the phase

The scalar product $k^{\mu}x_{\mu} = (\boldsymbol{k}.\boldsymbol{x} - \omega t)$ is a Lorentz invariant

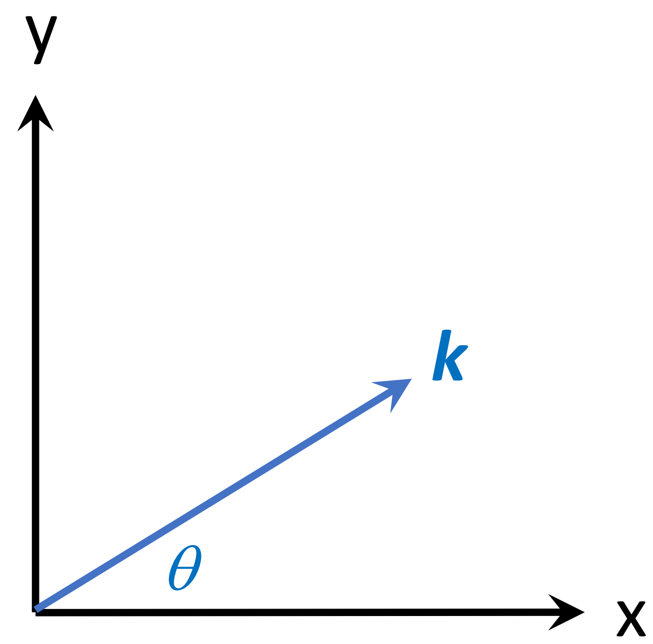
This is the phase of an EM wave: $\boldsymbol{E} \propto e^{\mathbf{i}(\boldsymbol{k}.\boldsymbol{x} - \ \omega t)}$

It makes sense that this should be Lorentz-invariant. A charge located at a place and time where E and B vanish will not accelerate, and all observers in an IRF need to agree about that. So they must all agree about where $(\boldsymbol{k}.\boldsymbol{x} - \ \omega t) = \left(n + \frac{1}{2}\right)\pi$

Doppler shift and aberration

Consider an EM wave propagating in the *xy*-plane at angle θ to the *x*-axis

$$k^{\mu} = \frac{\omega}{c}\begin{bmatrix} 1 \\ \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \frac{\omega}{c}\begin{bmatrix} 1 \\ \mu \\ (1-\mu^2)^{1/2} \\ 0 \end{bmatrix}$$



k^{μ} is a 4-vector ➔ we know how it transforms

$$k'^{\mu} = \Lambda^{\mu}{}_{\nu}\, k^{\nu}$$

Doppler shift and aberration

In a velocity boosted frame S' (v-boost in x-direction)

$$k'^{\mu} = \frac{\omega}{c}\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \mu \\ (1-\mu^2)^{1/2} \\ 0 \end{bmatrix} = \frac{\omega}{c}\begin{bmatrix} \gamma(1-\beta\mu) \\ \gamma(\mu-\beta) \\ (1-\mu^2)^{1/2} \\ 0 \end{bmatrix}$$

$$= \frac{\omega'}{c}\begin{bmatrix} 1 \\ \mu' \\ (1-\mu'^2)^{1/2} \\ 0 \end{bmatrix}$$

$$\omega' = \gamma(1-\beta\mu)\omega \quad \text{(Doppler shift)}$$

$$\mu' = \frac{\mu-\beta}{1-\beta\mu} \quad \text{(Aberration)}$$

Doppler shift and aberration

$$\omega' = \gamma(1 - \beta\mu)\omega \qquad\qquad \mu' = \frac{\mu - \beta}{1 - \beta\mu}$$

Limiting cases

$\mu = 1$: ($\theta = 0$ ➔ velocity boost along direction of ***k***)

$$\omega' = \gamma(1 - \beta)\,\omega = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega \sim \quad (1 - \beta)\omega \qquad \text{if } \beta \ll 1$$

$$\mu' = 1$$

$\mu = 0$: ($\theta = \pi/2$ ➔ velocity boost perpendicular to ***k***)

$$\omega' = \gamma\,\omega$$

$$\mu' = -\beta \quad \Rightarrow \sin\left(\frac{\pi}{2} - \theta'\right) = -\beta \quad \Rightarrow \theta' \sim \frac{\pi}{2} + \beta \quad \text{if } \beta \ll 1$$

(Note: for Earth's orbital motion around the Sun, β = 1.0 x 10^{-4} ⇒ $\theta' = 10^{-4}$ rad = 20'')

Differential power emitted by a relativistic particle

We are now in a position to compute the differential power, from an accelerating relativistic particle. We'll denote the instantaneous rest frame of the particle, S', and the observer frame (lab frame), S

Let's say S' is moving along the positive x-axis at speed *v,* and the differential power in the instantaneous rest frame is

$$\frac{dP'}{d\Omega'} = \frac{dE'}{d\Omega' dt'}$$

We previously computed the transformation from S to S',

$$\mu' = \frac{\mu - \beta}{1 - \beta\mu} \qquad \omega' = \gamma(1 - \beta\mu)\omega$$

Differential power emitted by a relativistic particle

So in the lab frame, we want to compute

$$\frac{dP}{d\Omega} = \frac{dE}{d\Omega dt} = \left(\frac{dE}{dE'}\right)\left(\frac{d\Omega'}{d\Omega}\right)\left(\frac{dt}{dt'}\right)^{-1}\frac{dE'}{d\Omega' dt'}$$

Let's consider these factors one at a time

$$\frac{dE}{dE'} = \frac{\omega}{\omega'} = \frac{1}{\gamma(1-\beta\mu)}$$

$$\frac{d\Omega'}{d\Omega} = \frac{d\mu'}{d\mu} = \frac{d}{d\mu}\left(\frac{\mu-\beta}{1-\beta\mu}\right) = \frac{(1-\beta\mu)+(\mu-\beta)\beta}{(1-\beta\mu)^2} = \frac{1-\beta^2}{(1-\beta\mu)^2} = \frac{1}{\gamma^2(1-\beta\mu)^2}$$

$$\frac{dt}{dt'} = \gamma$$

to obtain

$$\frac{dP}{d\Omega} = \frac{1}{\gamma^4(1-\beta\mu)^3}\frac{dP'}{d\Omega'}$$

Lecture 14
Special Relativity continued

Goals: understand

Relativistic beaming

Relativistic dynamics and the Lorentz force on a charge

Electromagnetism with 4-vectors

Emitted versus received power

This is an expression for the angular dependence of the *emitted* power, P_e

$$\frac{dP_e}{d\Omega} = \frac{1}{\gamma^4(1-\beta\mu)^3}\frac{dP'}{d\Omega'}$$

But this is different from the power RECEIVED by a stationary observer in the lab frame. If two photons are emitted at times $t_1{}'$ and $t_1{}' + dt\,'$, the difference between the ARRIVAL times will be $dt_A = \gamma dt\,'(1-\beta\mu)$

This is not the same as the difference between the emission times as determined in the lab frame S, $dt\ = \gamma dt'$, because of the difference in light travel times $\beta\mu\ dt$

So the power received has an additional factor of $(1-\beta\mu)$ in the denominator

$$\frac{dE}{d\Omega dt_A} = \frac{dP_r}{d\Omega} = \frac{dP_e}{d\Omega}\frac{dt}{dt_A} = \frac{1}{\gamma^4(1-\beta\mu)^{\mathbf{4}}}\frac{dP'}{d\Omega'}$$

Relativistic beaming

Let's consider first a source of radiation that is isotropic in its own rest frame

$$\frac{dP_r}{d\Omega} = \frac{1}{\gamma^4(1-\beta\mu)^4}\frac{P'}{4\pi}$$

For a highly relativistic particle with $\beta \sim 1$, the denominator becomes very small when $\mu = 1$ (i.e. when θ = 0) and the radiation is travelling along the positive x-axis (i.e. the direction of motion)

For θ = 0, $$\frac{dP_r}{d\Omega} = \frac{1}{\gamma^4(1-\beta)^4}\frac{P'}{4\pi}$$

Q1: In the limit $(1-\beta)$ << 1, how does $(1-\beta)$ depend on γ?

Relativistic beaming

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For θ = 0, $\frac{dP_r}{d\Omega} = \frac{1}{\gamma^4(1-\beta)^4}\frac{P'}{4\pi}$

Q1: In the limit $(1-\beta) << 1$, how does $(1-\beta)$ depend on γ?

A: $1/\gamma^2 = (1-\beta)\,(1+\beta) \sim 2(1-\beta)$ ➔ $(1-\beta) \sim 1/(2\gamma^{2})$

So $\frac{dP_r}{d\Omega} = 16\gamma^4\frac{P'}{4\pi}$ $\Rightarrow$ very strong beaming along the *x*-axis

Relation to the retarded potentials

Recall the κ factor in our expression for the Lienard-Wiechert potentials

$$\phi(\boldsymbol{x}, t) = \frac{q}{\kappa(t_{ret})R(t_{ret})} = \left[\frac{q}{\kappa R}\right]$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{q\boldsymbol{v}}{\kappa(t_{ret})R(t_{ret})} = \left[\frac{q\boldsymbol{v}}{c\kappa R}\right]$$

with $\kappa = (1 - \boldsymbol{v}.\hat{\boldsymbol{R}}/c)$

$1/\kappa \ \propto (1 - \beta)^{-1} \sim 2\gamma^2$

Hence, E and B are $\propto \gamma^2$ and $F \propto E \times B \propto \gamma^4$

Relativistic beaming

If we also take the limit of small θ as well as small as $1-\beta$, we may approximate μ by

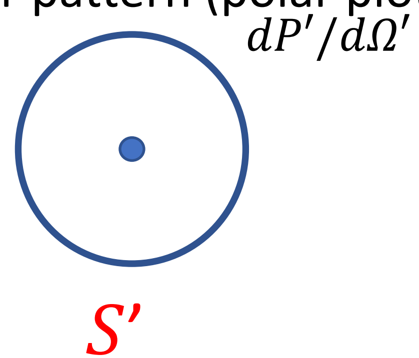
$(1-\theta^2/2)$ to obtain

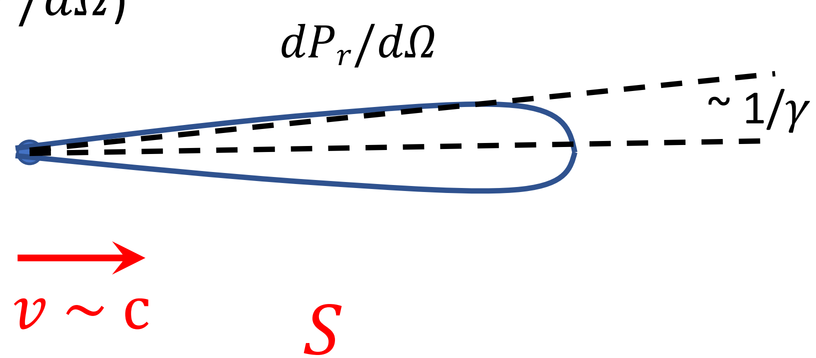
$$\frac{dP_r}{d\Omega} \sim \frac{1}{\gamma^4(1-\beta(1-\theta^2/2))^4}\frac{P'}{4\pi} \sim \frac{1}{\gamma^4(1-\beta+\theta^2/2)^4}\frac{P'}{4\pi}$$

$$\sim \frac{1}{\gamma^4(1/[2\gamma^2]+\theta^2/2))^4}\frac{P'}{4\pi} = \frac{16\gamma^4}{(1+\gamma^2\theta^2)^4}\frac{P'}{4\pi}$$

Thus the beam has an opening angle $\sim 1/\gamma$

Power pattern (polar plot with $r \propto dP/d\Omega$)





Accelerating, relativistic charge

In the instantaneous rest frame of an accelerating charge, the radiation is not isotropic but instead has a $\sin^2\Theta'$ dependence on the angle to the acceleration

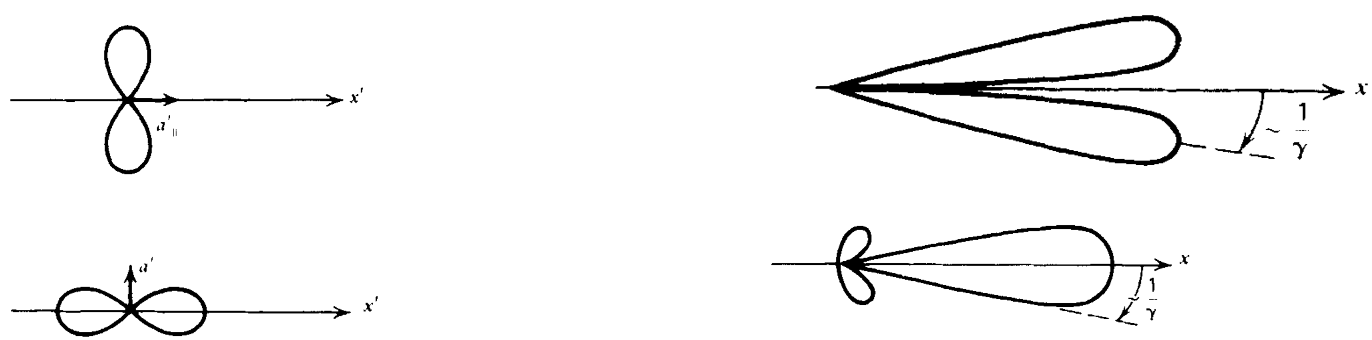
The power radiated is given by the Larmor formula, which may be written

$$dP'/d\Omega' = \frac{e^2|a'|^2\sin^2\Theta'}{c^3} = \frac{e^2 a^\mu a_\mu \sin^2\Theta'}{4\pi c^3}$$

The 3-acceleration may be at any angle to the 3-velocity, leading to a variety of beam patterns (*R&L* Fig 4.11)

$a' \parallel v$

$a' \perp v$



S' S

Four vector operators

For 3-vectors, a key vector operator is $\nabla \equiv \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$

For 4-vectors, the analog is $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu} = \begin{pmatrix} (1/c)\partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$

The analog of $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

is therefore $\partial_\mu \partial^\mu = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}$

This operator is Lorentz invariant and is called the d'Alembertian

It is variously written ∂^2, $\Box$, or $\Box^2$

Acceleration of a charge in an electric field

In 4-vector notation, we may write Newton's second law as

$$f^{\mu} = m_0\, a^{\mu}$$

Like a^{μ}, f^{μ} is orthogonal to U^{μ} , so in the instantaneous rest frame *S'*, where $U'^{\mu} = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}$, we must have

$$f'^{\mu} = \begin{pmatrix} 0 \\ \boldsymbol{f}' \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{qE}' \end{pmatrix}$$

But how do we know how the *B* and *E*-fields transform?

We'll need to formulate electromagnetism in a form that is Lorentz invariant, with equations involving 4-vectors and tensors.

The 4-current-density, j^μ

Let us define the 4-current-density as

$$j^\mu = \begin{pmatrix} \rho c \\ \boldsymbol{j} \end{pmatrix}$$

Q2: what is 4-divergence of j^μ i.e. what is $\partial_\mu j^\mu$

The 4-current-density, j^μ

Let us define the 4-current-density as

$$j^\mu = \begin{pmatrix} \rho c \\ \boldsymbol{j} \end{pmatrix}$$

Q2: what is $\partial_\mu j^\mu$

Answer: the 4-divergence of j^μ is zero

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{j} = 0 \qquad \text{by conservation of charge}$$

Because the right-hand-side is Lorentz-invariant and ∂_μ is a 4-vector operator, this shows that j^μ is indeed a 4-vector

The 4-potential, A^{μ}

We now observe that Maxwell's equations,

$$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t} = -4\pi\rho \quad \text{Gauss' Law}$$

$$\boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial^2 t} = -4\pi\boldsymbol{j}/\mathrm{c} \quad \text{Ampere's Law}$$

Can be written as a 4-vector equation

$$\partial_v\partial^v A^{\mu} = -\frac{4\pi j^{\mu}}{c}$$

where the 4-potential $A^{\mu} \equiv \begin{pmatrix}\phi \\ \boldsymbol{A}\end{pmatrix}$

The relation between ϕ and $\boldsymbol{A}$ for the Lorentz gauge we are using is also a 4-vector equation

$$\partial_{\mu}A^{\mu} = \frac{1}{c}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{A} = 0$$

The electric and magnetic fields

Since we know that A^{μ} transforms as a 4-vector, we can compute how

E and B transform

The fields can be treated very beautifully using this object that is a bit like the curl

$$F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}$$

This has 16 components (since α and β each take values from 0 to 3) and is a 2nd rank 4-tensor

It transforms according to $F_{\alpha\beta}{}' = \Lambda_{\alpha}^{\gamma}\,\Lambda_{\beta}^{\delta}\,F_{\gamma\delta}$

It is clearly antisymmetric, so there are six independent components (with zeros along the diagonal): amazingly, these components are just the E and B-fields (3 components for each)

The EM field tensor

$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is called the EM field tensor

Working out each component, we find that

$$F_{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

The Lorentz 4-force on a charge q with 4-velocity U^α is $f_\beta = \frac{1}{c} q F_{\alpha\beta} U^\alpha$

(We can confirm that in the instantaneous rest-frame, where $U'^\alpha = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}$,

$f'_\beta = \begin{pmatrix} 0 \\ q\boldsymbol{E}' \end{pmatrix}$ as required)

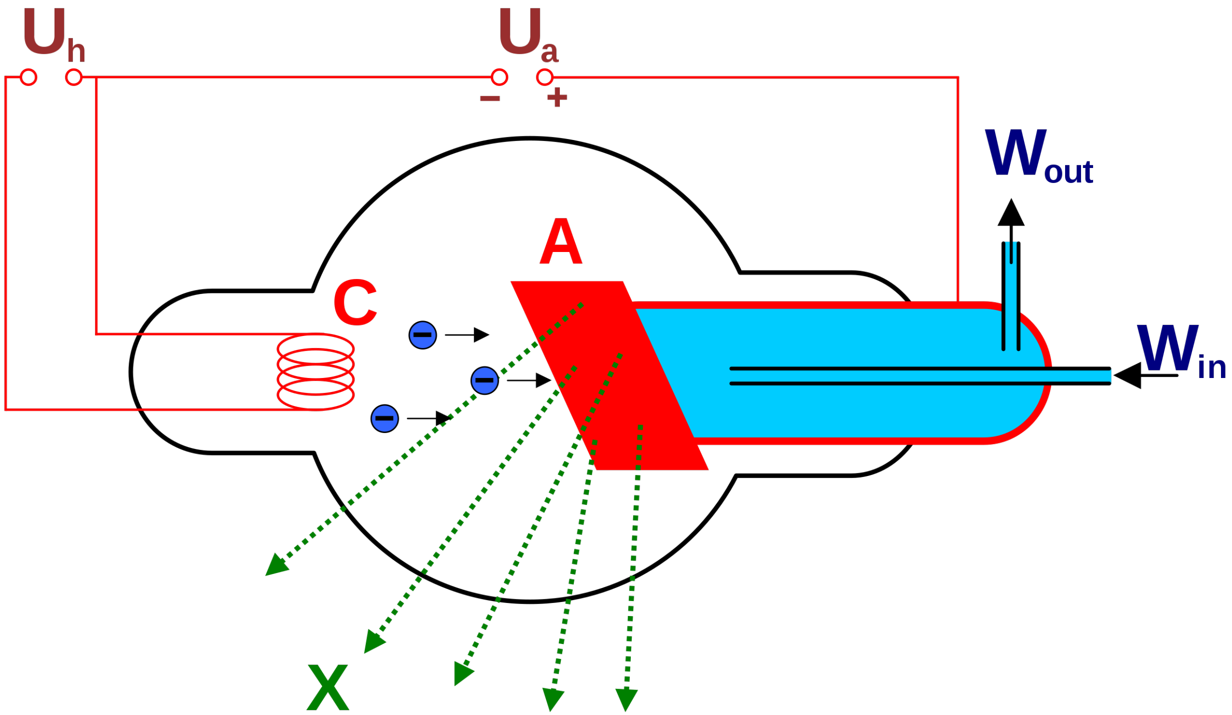
Summary: the Lorentz invariant laws of electromagnetism

Maxwell's Equations	$\partial_{\upsilon}\partial^{\upsilon}A^{\mu} = -\frac{4\pi j^{\mu}}{c}$
Lorentz Gauge	$\partial_{\mu}A^{\mu} = 0$
Conservation of charge	$\partial_{\mu}j^{\mu} = 0$
Definition of EM field tensor	$F_{\alpha\beta} \equiv \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$
Lorentz force	$f_{\beta} = \frac{1}{c}qF_{\alpha\beta}U^{\alpha}$

Bremsstrahlung: introduction

Bremsstrahlung = “braking radiation”

Example: X-ray tube (developed in the early 20th century)



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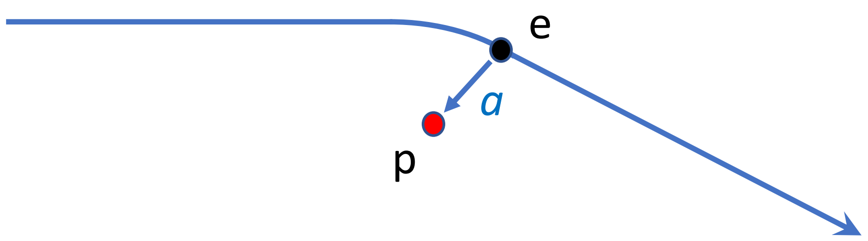
Bremsstrahlung: astrophysical context

In the astrophysical context, we are talking about the deflection of electrons in a plasma in close encounters with protons (or other ions)



Also known as *free-free* emission

Interstellar environments where we find a plasma include

1) Photoionized regions (HII regions, planetary nebulae

$$H + h\nu \rightarrow H^+ + e$$

2) Collisionally-ionized regions (behind shock waves)

$$H + e \rightarrow H^+ + e + e$$

HII regions and planetary nebulae

Gas is photoionized by a hot star with an effective temperature above $\sim 25{,}000$ K

Gas kinetic temperature $\sim 10^4$ K

Visible wavelength emission is dominated by spectral lines ("bound-bound" emission)

Orion nebula

Credits: NASA, ESA, M. Robberto and the Hubble Space Telescope Orion Treasury Project Team

Radio continuum emission (1.5 GHz map below by Subrahmanyan et al. 2001) is dominated by free-free emission

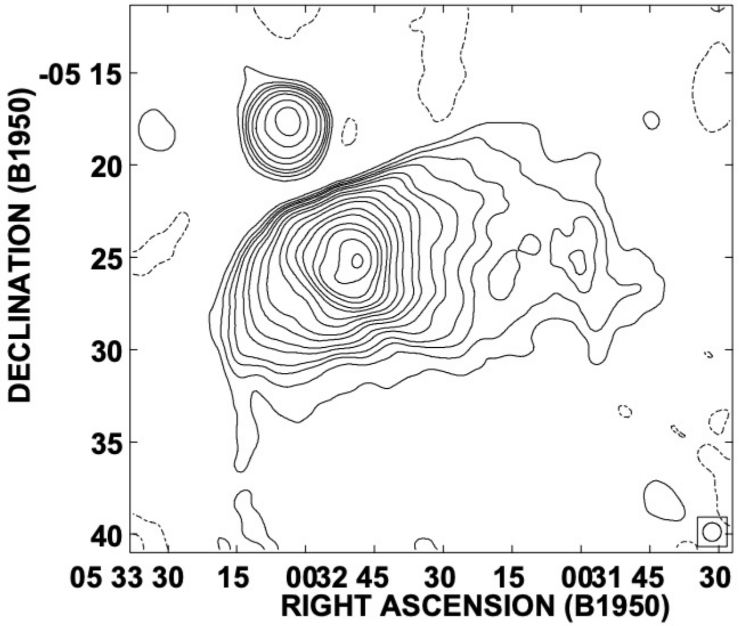


FIG. 4.—VLA 1.5 GHz image of the Orion region made with a beam of 1′ FWHM. Contours are at −0.1, 0.1, 0.2, 0.3, 0.4, 0.6, 0.8, 1.2, 1.6, 2.4, 3.2, 4.8, 6.4, 9.6, 12.8, 19.2, 25.6, and 38.4 Jy beam^{-1}. The image has been corrected for the primary beam attenuation.

Supernova shock waves

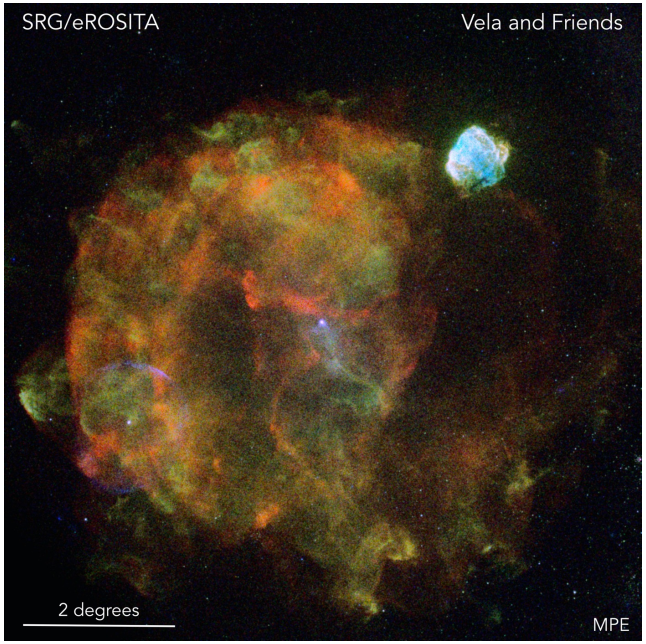
Supernovae release ~ 10^{51} erg of kinetic energy into the ISM, sending out expanding shock waves that can persist for tens of thousands of years

Gas kinetic temperature $\sim 10^6$ K or higher
The Vela SN remnant has an estimated age of 11,000 yr

Again, the visible wavelength emission is dominated by bound-bound transitions of ions

Filaments of the Vela Supernova Remnant
Image Credit & Copyright: Angus Lau, Y Van, SS Tong ([Jade Scope Observatory](#))

But free-free emission leads to radio and X-ray continuum (below)



Lecture 15

Bremsstrahlung

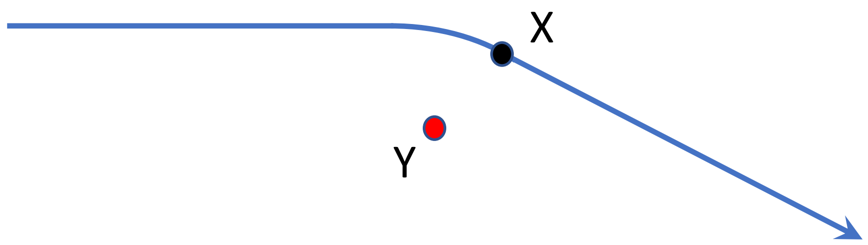
Goals: understand

Acceleration of electrons in collisions within a plasma

The significance of the b_{min} parameter

Emission from a (non-relativistic) collection of particles with a thermal distribution of velocities

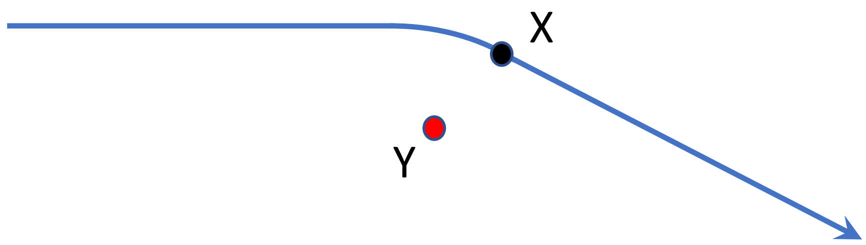
Acceleration of one free charge by another



Question 1: which of the following interactions can produce a non-zero $\ddot{d}$?

(a) e – p (b) e – H (c) e – e (d) $e^- - e^+$

Acceleration of one free charge by another



Question 1: which of the following interactions can produce a non-zero $\ddot{d}$?

(a) e – p (b) e – H (c) e – e (d) $e^- - e^+$

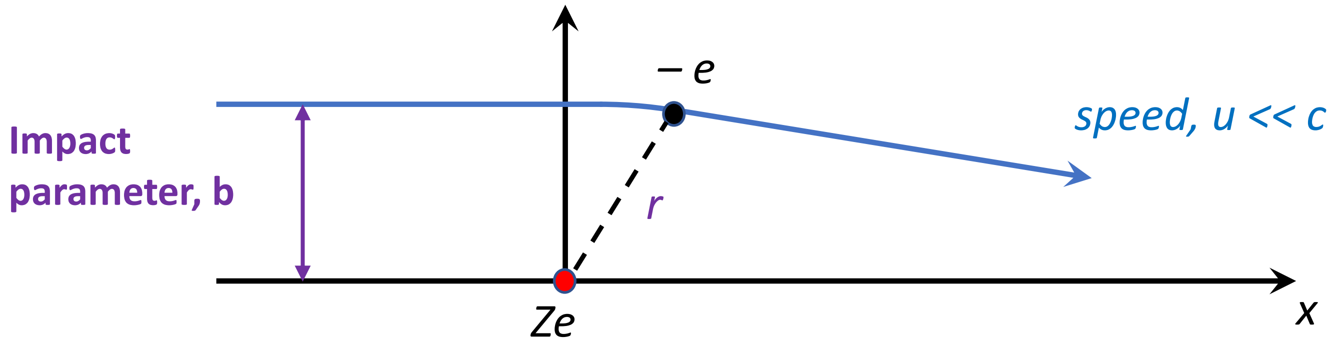
Answer: all except (c)

In an e – e collision, the center of charge remains at the center of mass ⇒ $\ddot{d} = 0$

Also, an e – H interaction can lead to a force on the electron, because the electron can induce a dipole moment in the atom

Here, we will focus on the first case, e – p (or more generally, e – ion)

Acceleration of one free charge by another



Let's consider an interaction at impact parameter b (not too small) so the deflection angle is small (i.e. << 1 rad)

In this limit, the *x* velocity is roughly constant, so $x = ut$
(if we define $t = 0$ as the moment of closest approach)

The separation, *r*, is well approximated by $\sqrt{b^2 + x^2} = \sqrt{b^2 + u^2t^2}$
and therefore the acceleration is $a = Ze^2/\,(m_e\, r^2) = Ze^2/(m_e[b^2 + u^2t^2])$

$$\Rightarrow \qquad |\ddot{d}| = ea = \frac{Ze^3}{m_e(b^2 + u^2t^2)}$$

Acceleration of one free charge by another

$$|\ddot{d}| = ea = \frac{Ze^3}{m_e(b^2 + u^2t^2)}$$

We can now compute the electric and magnetic fields at distance *R* in the wave zone

$$E^2 = B^2 = \frac{4\pi}{c} S = \frac{4\pi}{c} \frac{1}{R^2} \frac{dP}{d\Omega} = \frac{4\pi}{cR^2} \frac{|\ddot{d}|^2 \sin^2\Theta}{4\pi c^3}$$

$$\Rightarrow \qquad E(t) = \frac{|\ddot{d}| \sin\Theta}{c^2 R} = \frac{Ze^3 \sin\Theta}{m_e c^2 R(b^2 + u^2t^2)}$$

To get the spectrum of the emitted radiation, we are interested in the Fourier transform of the electric field (Lecture 8)

$$\tilde{E}_T(\omega) = \frac{1}{2\pi}\int_{-T/2}^{T/2} e^{i\omega t} E(t)dt \to \frac{Ze^3 \sin\Theta}{2\pi m_e c^2 R b^2}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{1 + t^2/t_c^2} dt \quad as\, T \to \infty$$

where $t_c = b/u$ is the “collision time”

Acceleration of one free charge by another

The integral

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{1 + t^2/t_c^2}\, dt \quad = \pi t_c e^{-\omega t_c}$$

$$\Rightarrow \quad \tilde{E}_T(\omega) = \frac{Ze^3 \sin\Theta}{m_e c^2 R b^2 t_c} e^{-\omega t_c} = \frac{Ze^3 \sin\Theta}{m_e c^2 R b u} e^{-\omega b/u}$$

The average monochromatic flux at distance *R* is then (Lecture 8)

$$F_\nu = \frac{2\pi c}{T} |\tilde{E}_T(\omega)|^2 = \frac{\pi Z^2 e^6 \sin^2\Theta}{2 m_e^2 c^3 R^2 T b^2 u^2} e^{-4\pi\nu b/u}$$

(average during time period *T*)

The radiant energy at frequency ν emitted due to this interaction is then

$$W_\nu = T \int F_\nu dA = T \int R^2 F_\nu d\Omega = \frac{\pi Z^2 e^6\, 8\pi/3}{2 m_e^2 c^3 b^2 u^2} e^{-2\omega b/u} = \frac{4 Z^2 e^6}{3 m_e^2 c^3} \left(\frac{\pi}{bu}\right)^2 e^{-4\pi\nu b/u}$$

Key points

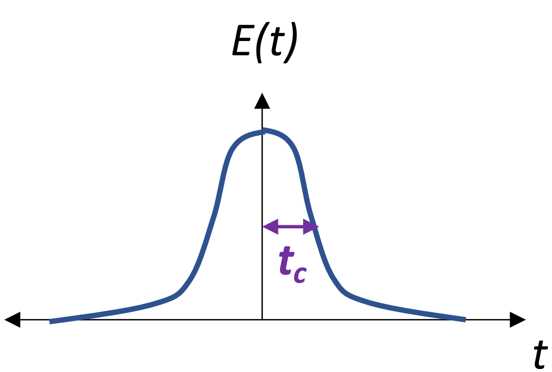
$$W_\upsilon = \frac{4Z^2e^6}{3m_e^2c^3}\left(\frac{\pi}{bu}\right)^2 e^{-4\pi\nu b/u}$$

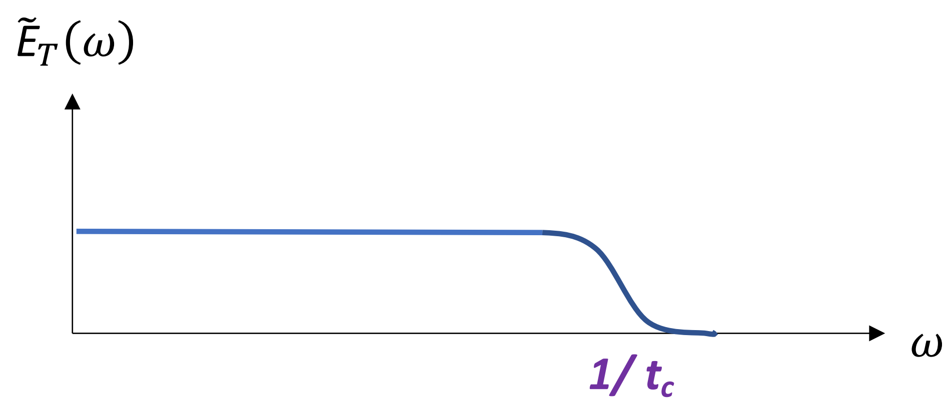
This expression is the amount of energy emitted per unit bandwidth, *dE/d*ν, due to a single collision with impact parameter *b* and electron velocity *u*

Key approximations:

(1) we have neglected quantum effects: this is a purely "classical result"
(2) we neglected the acceleration of the ion
(3) we assumed a small angle deflection

Key feature: the distribution in frequency is nearly flat until $\omega t_c = \omega b/u$ reaches ~ 1





Emission from an ensemble of particles

Let's focus on a single ion, and think about all the electrons that may hit it

Rate at which electrons hit at impact parameter between b and $b + db$

$$= n_e u \, dA = n_e u \, 2\pi b \, db$$

Monochromatic power emitted in such collisions

$$= W_\upsilon \, n_e u \, 2\pi b \, db = \frac{4Z^2 e^6}{3m_e^2 c^3} \left(\frac{\pi}{bu}\right)^2 e^{-4\pi\nu b/u} \, n_e u \, 2\pi b \, db$$

$$= \frac{8\pi^3 Z^2 e^6 n_e}{3m_e^2 c^3 u} \frac{db}{b} e^{-4\pi\upsilon b/u}$$

Total power per ion due to all collisions $= \frac{8\pi^3 Z^2 e^6 n_e}{3m_e^2 c^3 u} \int \frac{db}{b} e^{-4\pi\upsilon b/u}$

Problem: the integral diverges (but only logarithmically) at small *b,* implying that the power radiated is infinite

What went wrong?

Two approximations that we made break down at small b

(1) Our approximation that the deflection angle is small.

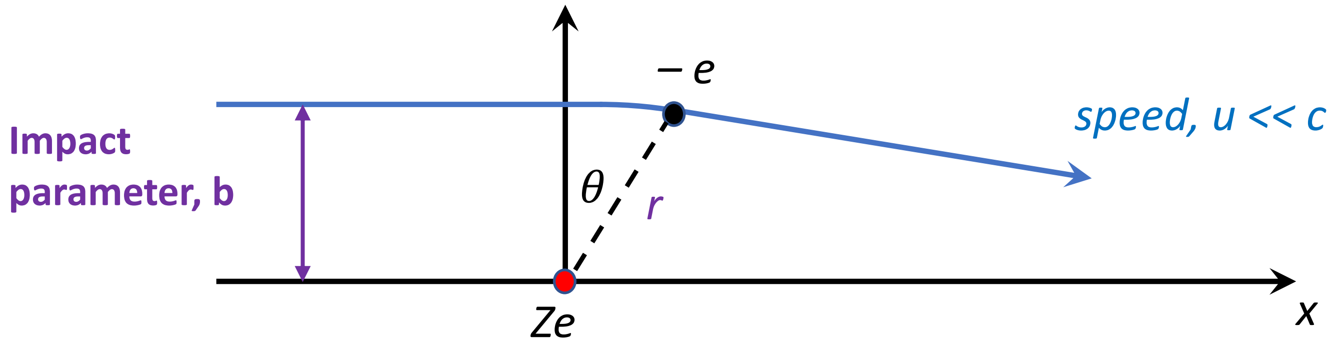
(2) Our neglect of quantum effects, as the angular momentum $m_e\, b\, u$ is quantized in units of $\hbar$

Since the divergence of the integral is logarithmic, even a rough estimate of where the approximations break down can yield a useful result.

So the idea is to truncate the integral $\int \frac{db}{b} e^{-4\pi \nu b/u}$ at some lower limit, b_{min}, where the approximations tend to break down.

Let’s consider each of them in turn.

Small deflection angle approximation



Let us consider the acceleration in the y-direction (i.e. perpendicular to the initial direction of motion)

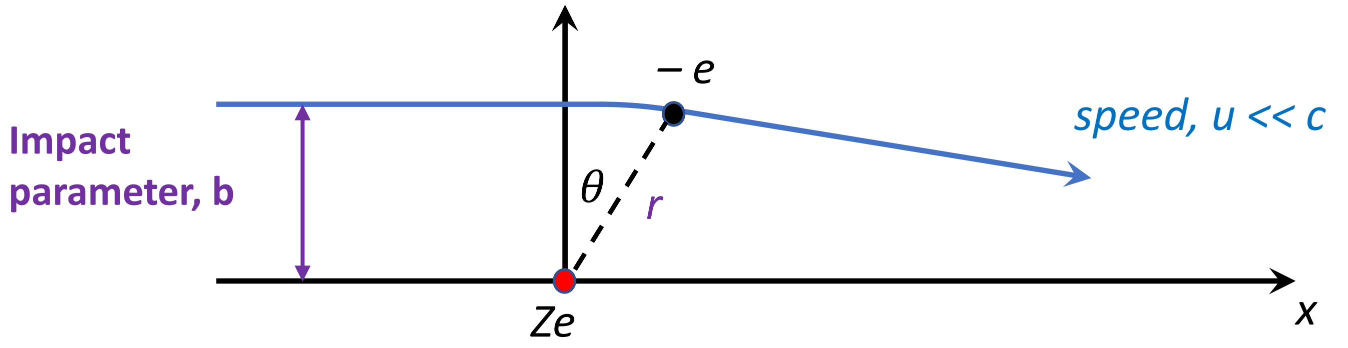
$$a_y = -a\cos\theta = -a\frac{b}{r} = -\frac{Ze^2}{m_e r^2}\frac{b}{r} = -\frac{Ze^2 b}{m_e(b^2+u^2t^2)^{3/2}}$$

The *y*-velocity the electron acquires during the collision is

$$u_y = \int_{-\infty}^{\infty} a_y\,dt = -\frac{Ze^2}{m_e u b}\int_{-\infty}^{\infty}\frac{dq}{(1+q^2)^{\frac{3}{2}}} = -\frac{2Ze^2}{m_e u b}$$

where $q = ut/b$

Small deflection angle approximation



The deflection angle is small if and only if

$$|u_y| = \frac{2Ze^2}{m_e u b} \ll u$$

or equivalently

$$b \gg \frac{Ze^2}{m_e u^2/2}$$

We'll call the right-hand-side of this inequality $b_{min}^{(1)}$

The small angle approximation breaks down for b smaller than $\sim b_{min}^{(1)}$

Classical treatment of electron motion

The classical treatment of the electron motion is valid if and only if

Electron angular momentum

$$l = m_e u b \gg \hbar$$

or equivalently

$$b \gg \frac{\hbar}{m_e u}$$

We'll call the right-hand-side of this inequality $b_{min}^{(2)}$

The classical treatment of the electron motion breaks down for b smaller than $\sim b_{min}^{(2)}$

Bottom line: our treatment is valid if and only if $b \gg b_{min}^{(1)}$ and $b \gg b_{min}^{(2)}$

Take total power per ion due to all collisions $= \frac{8\pi^3 Z^2 e^6 n_e}{3 m_e^2 c^3 u} \int_{b_{min}}^{\infty} \frac{db}{b} e^{-4\pi \upsilon b/u}$

where b_{min} is the *larger* of $b_{min}^{(1)}$ and $b_{min}^{(2)}$

Which is larger, $b_{min}^{(1)}$ or $b_{min}^{(2)}$?

Answer: it depends on u

$$b_{min}^{(1)} = \frac{Ze^2}{m_e u^2/2} \qquad b_{min}^{(2)} = \frac{\hbar}{m_e u}$$

so

$$\frac{b_{min}^{(2)}}{b_{min}^{(1)}} = \frac{u\hbar}{2Ze^2} = \frac{1}{2}\left(\frac{K_e}{\chi}\right)^{1/2}$$

where $K_e = m_e u^2/2$ is the electron kinetic energy, and

$\chi = Z^2 m_e e^4/[2\hbar^2]$= 13.6 Z^2 eV is the ionization potential of a H-like ion of charge Ze

So, for K_e larger than χ, $b_{min}^{(2)}$ is larger and the classical treatment of the electron motion breaks down first as $b \rightarrow 0$

So, for K_e smaller than χ, $b_{min}^{(1)}$ is larger and the small angle approximation breaks down first as $b \rightarrow 0$

Which is larger, $b_{min}^{(1)}$ or $b_{min}^{(2)}$?

In an HII region, $T \sim 10^4$ K

$\Rightarrow$ typical electron K. E. $\sim kT \sim 1$ eV $\ll \chi = 13.6$ eV (for H)

$\Rightarrow b_{min}^{(1)} > b_{min}^{(2)} \Rightarrow$ small angle approximation is what breaks down

In a hot shocked region, $T \sim 106$ K and the opposite is usually true

Power emitted per ion

In any case, our integral $\int_{b_{min}}^{\infty} \frac{db}{b} e^{-4\pi \upsilon b/u}$ may be written $\int_{\xi_{min}}^{\infty} \frac{e^{-\xi}}{\xi} d\xi \equiv E_1(\xi_{min})$

where E_1 is the exponential integral function and $\xi_{min} = \frac{4\pi b_{min}\upsilon}{u}$

The total power per ion due to all collisions = $\frac{8\pi^3 Z^2 e^6 n_e}{3m_e^2 c^3 u} E_1(\xi_{min})$

For monoenergetic electrons at velocity u, we can determine the emission coefficient by multiplying by the density of ions n_i and dividing by 4π

$$j_\nu^{\mathrm{ff}}(u) = \frac{2\pi^2 Z^2 e^6 n_e n_i}{3m_e^2 c^3 u} E_1(\xi_{min})$$

Because of the quantization of photon energies, this expression is only correct when $h\nu < K_e$. If that condition does not apply, then $j_\nu^{\mathrm{ff}}(u) = 0$

Lecture 16
Bremsstrahlung/ introduction to synchrotron radiation

Goals: understand

Emission from a (non-relativistic) collection of particles with a thermal distribution of velocities

Free-free *absorption*

Astrophysical introduction to cosmic rays and synchrotron radiation

Emission coefficient for a thermal distribution of electron energies

We can average
$$j_\nu^{\rm ff}(u) = \frac{2\pi^2 Z^2 e^6 n_e n_i}{3m_e^2 c^3 u} E_1(\xi_{min}) \quad \text{for } \nu < \frac{m_e u^2}{2h}$$
$$= 0 \quad \text{for } \nu > \frac{m_e u^2}{2h}$$
over a Maxwell-Boltzmann distribution of electron velocities to obtain the overall emission coefficient for a plasma at temperature T

I'll omit the details, but neglecting the weak logarithmic factor involving E_1, the temperature and frequency dependence must be $j_\nu^{\rm ff}(T) \propto T^{-1/2} \exp(-\frac{h\nu}{kT})$

We end up with $j_\nu^{\rm ff}(T) = \frac{16 Z^2 e^6}{3 m_e c^3}\left(\frac{2\pi}{3kTm_e}\right)^{1/2} n_e n_i \, \bar{g}_{\rm ff} \exp(-\frac{h\nu}{kT})$

where $\bar{g}_{\rm ff}(\nu, T)$ is a fudge factor (the "Gaunt factor") of order unity

Emission coefficient for a thermal distribution of electron energies

In cgs units, we get

$$\frac{j_\nu^{\mathrm{ff}}(T)}{\mathrm{erg\,cm^{-3}s^{-1}sr^{-1}Hz^{-1}}} = 1.1\times10^{-38}\left(\frac{n_e}{\mathrm{cm^{-3}}}\right)\left(\frac{n_i}{\mathrm{cm^{-3}}}\right)\bar{g}_{\mathrm{ff}}\,Z^2\left(\frac{T}{\mathrm{K}}\right)^{-1/2}\mathrm{e}^{-h\nu/kT}$$

A proper treatment, with careful inclusion of quantum mechanical effects, is needed to compute the Gaunt factor.

$\bar{g}_{\mathrm{ff}}(\nu, T)$ decreases from ~ 5 at $h\nu/kT$ = 10^{-4} to ~ 1 at $h\nu/kT$ = 1 (see R&L figure 5.3)

Integrating over frequency and solid angle, we get the power emitted per unit volume:

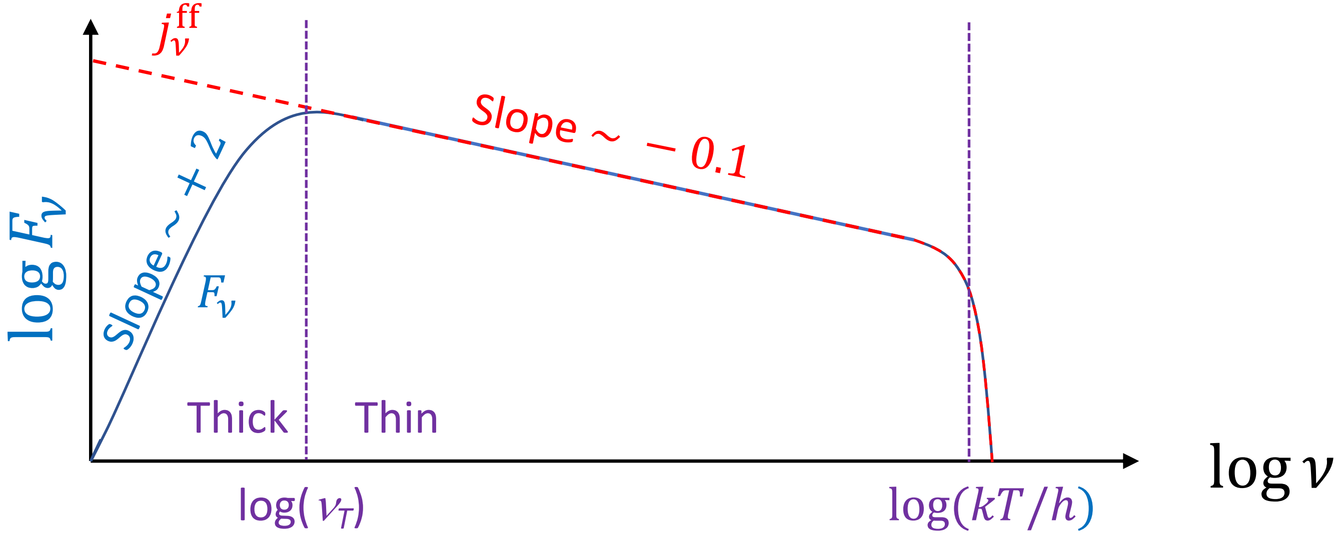
$$\frac{\int 4\pi j_\nu^{\mathrm{ff}}(T)\,d\nu}{\mathrm{erg\,cm^{-3}s^{-1}}} = 1.4\times10^{-27}\left(\frac{n_e}{\mathrm{cm^{-3}}}\right)\left(\frac{n_i}{\mathrm{cm^{-3}}}\right)\bar{g}_{\mathrm{B}}\left(\frac{T}{\mathrm{K}}\right)^{1/2}Z^2$$

Here the frequency-averaged Gaunt factor $\bar{g}_{\mathrm{B}}(T)\sim 1.2\ \pm 20\%$

Observed spectra of HII regions

At high frequencies, the free-free continuum flux measured from HII regions, F_ν is usually in good agreement with j_ν^{ff}

But at low energies, the flux typically drops at low frequency



This behavior suggests that optical depth effects become important at low frequency ($\nu < \nu_T$) so I_ν approaches the Planck function and $F_\nu \propto \nu^2$ (R-J limit)

Free-free absorption

Free-free absorption is the inverse process to free-free emission: radiation can be absorbed during a collision

Because we have a thermal plasma, we can use *Kirchhoff's Law* to compute the absorption coefficient

$$\alpha_\nu^{\rm ff} = \frac{j_\nu^{\rm ff}}{B_\nu} \propto \frac{\lambda^2}{T} j_\nu^{\rm ff} \propto \lambda^2\, T^{-3/2} Z^2 n_e n_i\, \bar{g}_{\rm ff} \qquad (\text{for } hc/\lambda \gg kT)$$

The optical depth $\tau_\nu^{\rm ff} = \int \alpha_\nu^{\rm ff} ds \propto \int n_e n_i\, ds \equiv$ "Emission measure", EM

$$\tau_\nu^{\rm ff} = 6.2 \times 10^{-11} Z^2 \left(\frac{T}{10^4 \mathrm{K}}\right)^{-3/2} \left(\frac{\lambda}{\mathrm{cm}}\right)^2 \bar{g}_{\rm ff} \frac{EM}{\mathrm{cm}^{-6}\mathrm{pc}}$$

For the Orion nebula, $T \sim 10^4 \mathrm{K}$ and $EM \sim 10^6\ \mathrm{cm}^{-6}\mathrm{pc}$
($n_e \sim n_i \sim 10^3\ \mathrm{cm}^{-3}$ and $D \sim 1$ pc)

$\tau_\nu^{\rm ff} \sim 1$ at $\lambda \sim 50$ cm (where $\bar{g}_{\rm ff} \sim 5$) or equivalently $\nu \sim 0.6$ GHz

Energetic charged particles in astrophysics

Energetic charged particles are well known in the Milky Way (directly detectable as "cosmic rays") and in external galaxies, especially radiogalaxies

Discovery of cosmic rays by Victor Hess

Victor F. Hess, center, departing from Vienna about 1911, was awarded the Nobel Prize in Physics in 1936. (New York Times, August 7, 2012, page D4)

The nature of CR was controversional


"All the News That's Fit to Print."

The New York Times.

LATE CITY EDITION

NEW YORK, SATURDAY, DECEMBER 31, 1932.

MILLIKAN RETORTS HOTLY TO COMPTON IN COSMIC RAY CLASH

ROOSEVELT WORKS WITH HOUSE CHIEFS FOR FEDERAL CUTS

Mussolini Will Curb Industrial Output As a Means of Combating Depression

O'BRIEN TO EXTEND REFORM PROGRAM; TAKES OATH TODAY

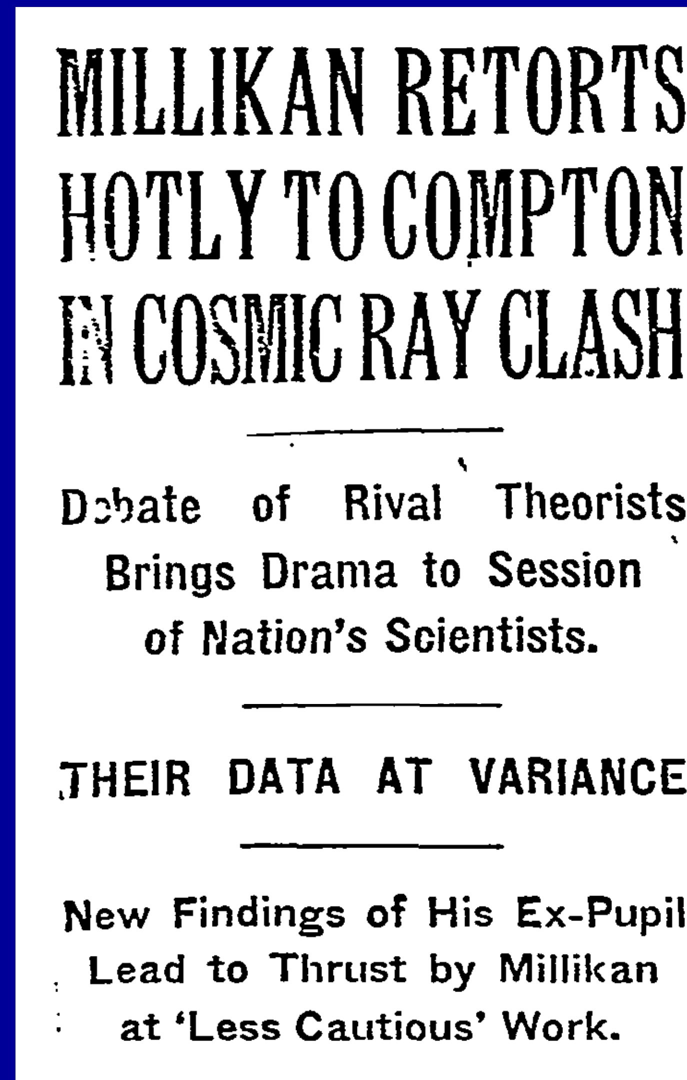
LEHMAN TAKES OATH PRIVATELY TONIGHT IN CEREMONY HERE

REVELRY TO SPEED OLD YEAR, HAIL NEW

AMAZON WAR ZONE TOUCHES 4 NATIONS

WINTER QUITS POST IN ANTI-KOENIG ROW

HOOVER HITS LUCK, GETS 7-FOOT FISH


MILLIKAN RETORTS HOTLY TO COMPTON IN COSMIC RAY CLASH

Debate of Rival Theorists Brings Drama to Session of Nation's Scientists.

THEIR DATA AT VARIANCE

New Findings of His Ex-Pupil Lead to Thrust by Millikan at 'Less Cautious' Work.

The nature of CR was controversional

By WILLIAM L. LAURENCE.
Special to THE NEW YORK TIMES.

ATLANTIC CITY, Dec. 30.—Professor Robert A. Millikan, who won the Nobel Prize in physics for being the first to measure the charge of the electron, and his former pupil, Professor Arthur H. Compton, who won the Nobel Prize for the discovery of the Compton effect, presented today before the American Association for the Advancement of Science two diametrically opposed hypotheses on the nature of the cosmic ray, in the study of which Dr. Millikan is the American pioneer.

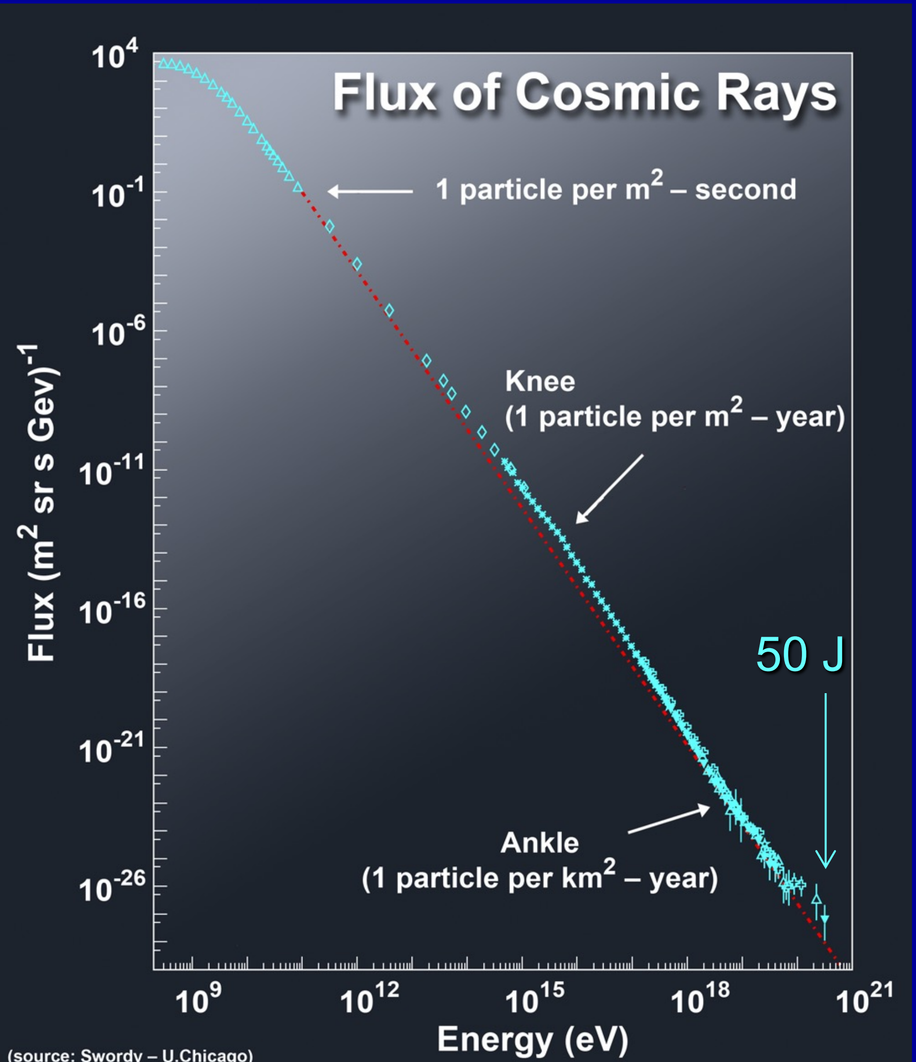
In an atmosphere surcharged with drama, in which the human element was by no means lacking, the two protagonists presented their views with the vehemence and fervor of those theoretical debates of bygone days when learned men clashed over the number of angels that could dance on the point of a needle. Dr. Millikan particularly sprinkled his talk with remarks directly aimed at his antagonist's scientific acumen. There was obvious coolness between the two men when they met after the debate was over.

Dr. Millikan holds that the cosmic rays are photons, bullets of light, similar in nature to the gamma-rays from radium. Such rays have no electric charge, travel with the speed of light, and are wave-like in nature. Professor Compton, on the other hand, holds that the cosmic rays are electrons, electrically charged particles, of the same nature as the ultimate units of matter, one of the two kinds of "bricks" out of which all the elements in the universe are constituted. These diametrically opposed hypotheses are largely based on diametrically opposed findings of fact.

In upholding the photon theory of the cosmic ray, Dr. Millikan was at the same time championing another cause, though he did not mention it today. To him the cosmic ray "furnishes some experimental evidence that the Creator is still on the job," that the rays are really "birth cries" of new matter being constantly replenished in the interstellar spaces. But this hypothesis is closely bound up with the assumption that the cosmic rays are photons. If they are electrons, such a hypothesis of new creation becomes untenable.

Cosmic ray energy spectrum

CR are observed over a remarkable range of energies



Cosmic ray spectrum (credit: HAP / A. Chantelauze)

Maximum energy detected to date: 50 J
(~ K.E. of a fastball in Little League baseball)

Total energy density ~ 1 eV cm^{-3}

… somewhat LARGER than that of starlight, the CMB, or the Galactic B-field

Table 1.5 Energy Densities in the Local ISM

Component	$u(\mathrm{eV\,cm^{-3}})$	Note
Cosmic microwave background ($T_{\mathrm{CMB}} = 2.725\,\mathrm{K}$)	0.265	a
Far-infrared radiation from dust	0.31	b
Starlight ($h\nu < 13.6\,\mathrm{eV}$)	0.54	c
Thermal kinetic energy $(3/2)nkT$	0.49	d
Turbulent kinetic energy $(1/2)\rho v^2$	0.22	e
Magnetic energy $B^2/8\pi$	0.89	f
Cosmic rays	1.39	g

a Fixsen & Mather (2002).
b Chapter 12.
c Chapter 12.
d For $nT = 3800\,\mathrm{cm^{-3}\,K}$ (see §17.7).
e For $n_{\mathrm{H}} = 30\,\mathrm{cm^{-3}}$, $v = 1\,\mathrm{km\,s^{-1}}$, or $\langle n_{\mathrm{H}}\rangle = 1\,\mathrm{cm^{-3}}$, $\langle v^2\rangle^{1/2} = 5.5\,\mathrm{km\,s^{-1}}$.
f For median $B_{\mathrm{tot}} \approx 6.0\,\mu\mathrm{G}$ (Heiles & Crutcher 2005).
g For cosmic ray spectrum X3 in Fig. 13.5.

Draine, 2011

Interaction with the interstellar gas

- High energy (E > 280 MeV) cosmic rays create γ-rays via

  $CRp + p \rightarrow CRp + p + \pi^0$

  $\pi^0 \rightarrow \gamma + \gamma$

- Lower energy cosmic rays ionize and heat the ISM

  $CRp + H \rightarrow CRp + H^+ + e$

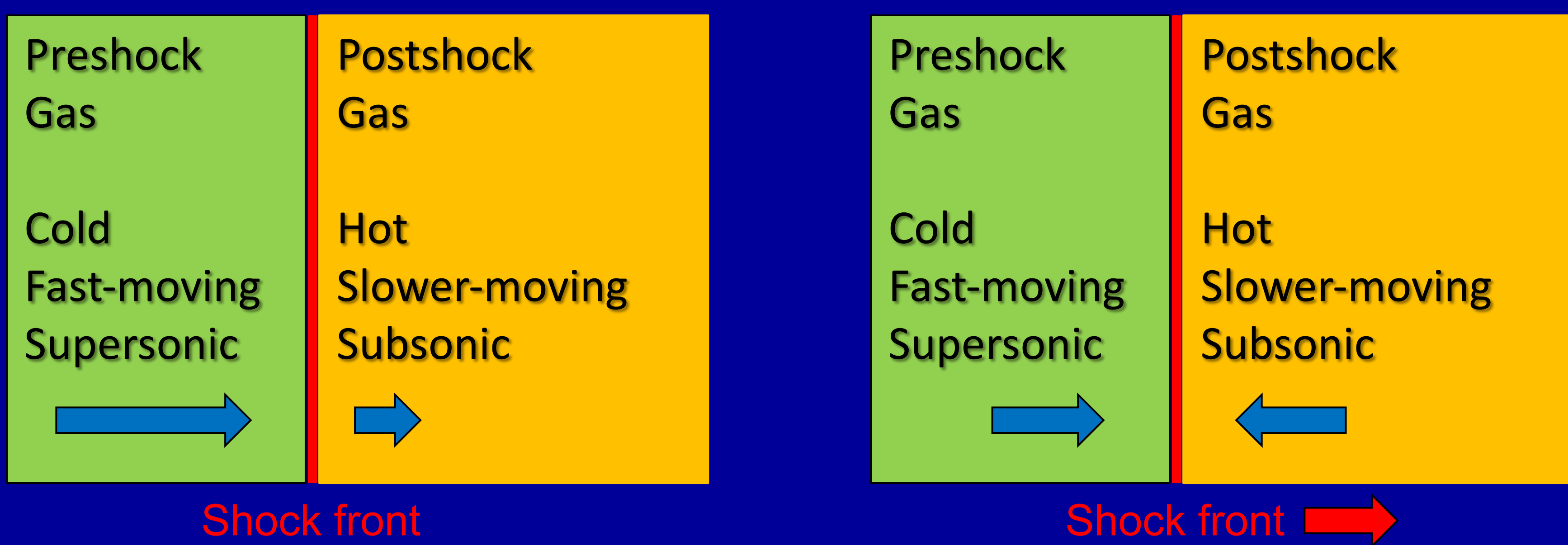
  $CRp + H_2 \rightarrow CRp + H_2^+ + e$

- Secondary electrons can cause additional ionization and heating, and can excite UV emissions from H and H_2 (important in dense clouds where starlight is absent)

Origin of cosmic rays

Primary origin is believed to be fast shocks in supernova remnants

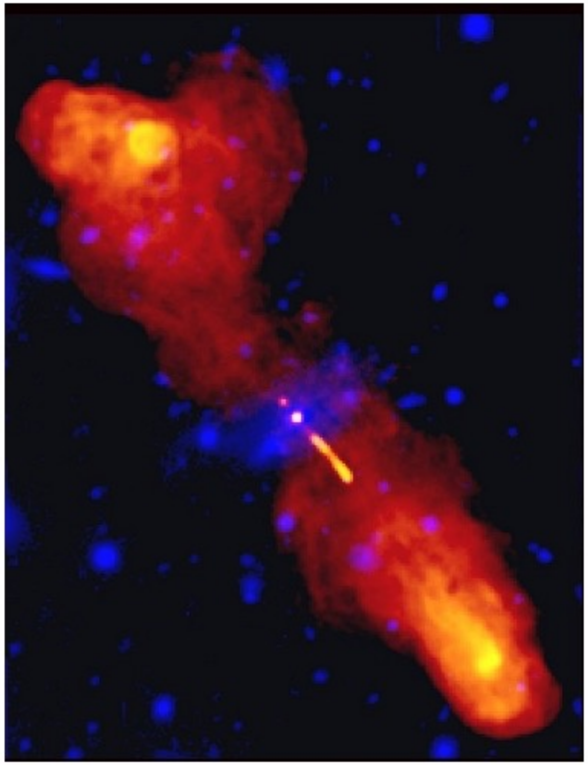
Fermi acceleration of particles "trapped" between converging "magnetic" mirrors (Fermi 1954)



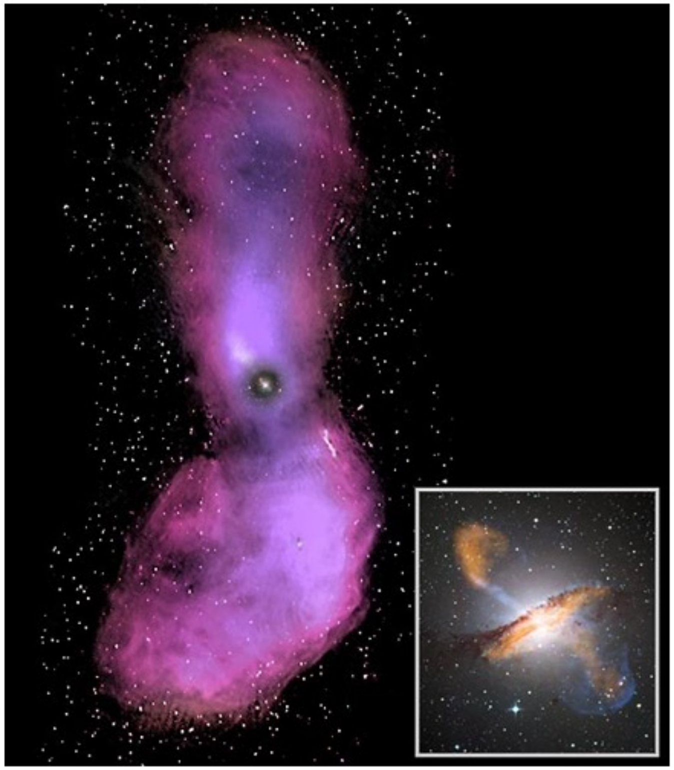
Radio galaxies

Active galaxies can produce giant jets of relativistic electrons that we can detect via synchrotron radiation

3C 237 (*size* = 290 kpc)



Centaurus A (*size* = 700 kpc)



Credits: NRAO, CSIRO/ATNF, ATCA, ASTRON, Parkes, MPIfR, ESO/WFI/AAO (UKST), MPIfR/ESO/APEX, NASA/CXC/CfA

Lecture 17
Synchrotron radiation I

Goals: understand

Motion of a relativistic electron in a magnetic field

Power radiated by such an electron

The spectrum of synchrotron radiation (for an electron with a specific energy)

Electron equation of motion

The electron equation of motion is

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} F^\mu{}_\nu U^\mu$$

$$\Rightarrow \; m_e \frac{d}{d\tau}\begin{pmatrix}\gamma c \\ \gamma \boldsymbol{v}\end{pmatrix} = \frac{q}{c}\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}\begin{bmatrix}\gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z\end{bmatrix} = \frac{q}{c}\gamma\begin{bmatrix}\boldsymbol{v}.\boldsymbol{E} \\ c\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\end{bmatrix}$$

If E is zero in the lab frame, we find

$$m_e \cancel{\gamma} \frac{d}{dt}\begin{bmatrix}\gamma c \\ \gamma \boldsymbol{v}\end{bmatrix} = \frac{q}{c}\cancel{\gamma}\begin{bmatrix}\boldsymbol{0} \\ \boldsymbol{v}\times\boldsymbol{B}\end{bmatrix} \Rightarrow \; m_e \gamma \frac{d\boldsymbol{v}}{dt} = \frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}$$

and $\gamma = \text{constant}$ $\quad\Rightarrow\quad$ $|\boldsymbol{v}| = \text{constant}$

Electron equation of motion

$$m_e \gamma \frac{d\boldsymbol{v}}{dt} = \frac{q}{c} \boldsymbol{v} \times \boldsymbol{B} \qquad\qquad |\boldsymbol{v}| = \text{constant}$$

We can decompose v into the components parallel and perpendicular to the B-field

$$\boldsymbol{v} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}$$

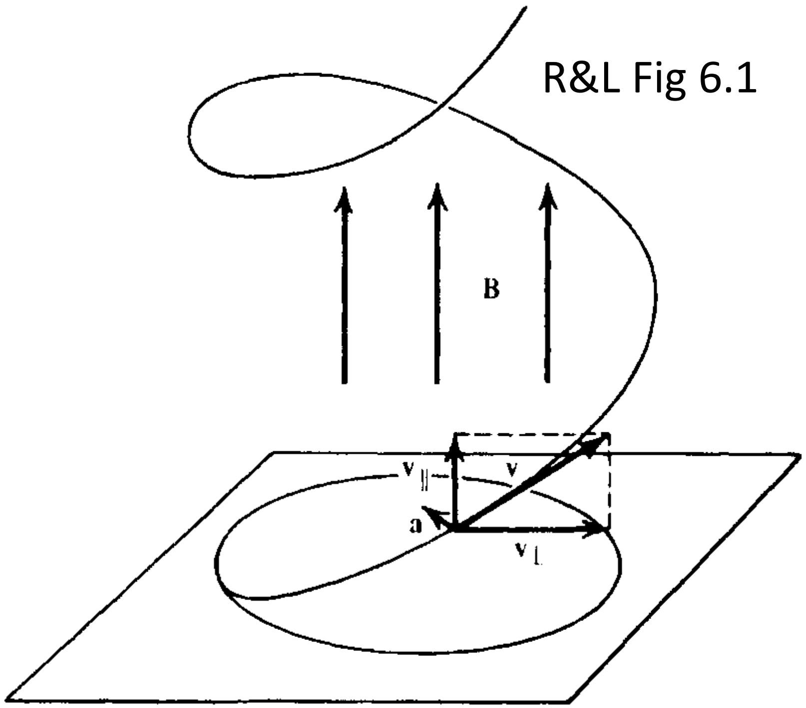
and write

$$\frac{d\boldsymbol{v}_{\parallel}}{dt} = 0 \qquad\qquad \frac{d\boldsymbol{v}_{\perp}}{dt} = \frac{q}{m_e \gamma c} \boldsymbol{v}_{\perp} \times \boldsymbol{B}$$

Thus,

$$\boldsymbol{v}_{\parallel} = \text{constant} \;\; \text{and} \;\; |\boldsymbol{v}_{\perp}| = \text{constant}$$

R&L Fig 6.1

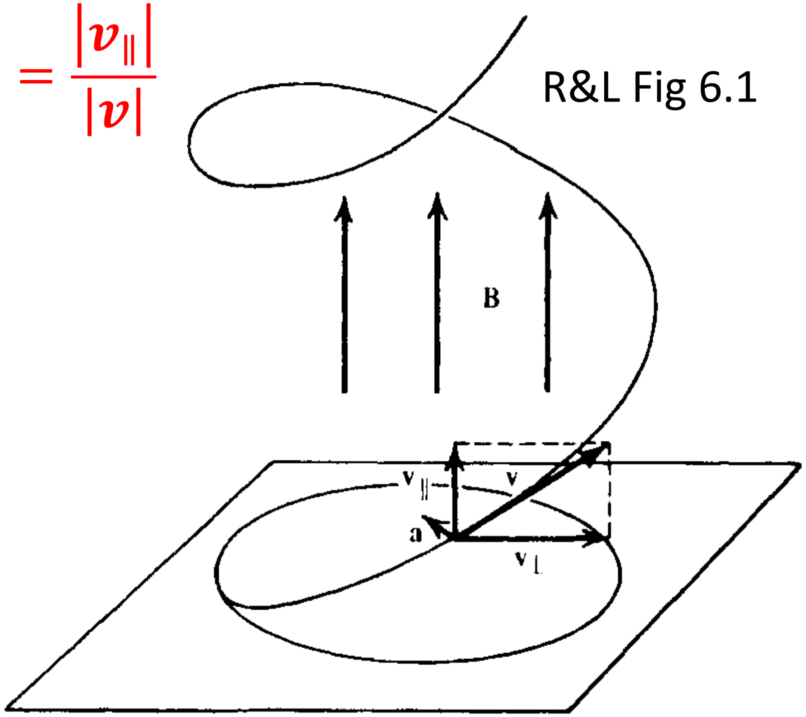


The electron executes a helical motion at constant speed

Pitch angle

We define the pitch angle, α, as the angle between $\boldsymbol{v}$ and $\boldsymbol{B}$

$$\sin\alpha = \frac{|\boldsymbol{v}_\perp|}{|\boldsymbol{v}|} \qquad \cos\alpha = \frac{|\boldsymbol{v}_\parallel|}{|\boldsymbol{v}|}$$



R&L Fig 6.1

so $\alpha = \pi/2$ for circular motion $\perp$ to $\boldsymbol{B}$

$\alpha = 0$ for linear motion along $\boldsymbol{B}$

Larmor frequency

Angular frequency of gyration, ω_B, is derived from

$$a = \left|\frac{d\boldsymbol{v}_\perp}{dt}\right| = \frac{|\boldsymbol{v}_\perp|^2}{r} = v_\perp \omega_B$$

$$\frac{q}{m_e \gamma c} v_\perp B = v_\perp \omega_B$$

$$\Rightarrow \ \omega_B = \frac{qB}{m_e \gamma c} \quad \text{This is called the Larmor frequency}$$

The Larmor frequency is independent of the pitch angle and depends on v only through γ

For an electron, $\omega_B = 17.6 \frac{B}{\mu G} \gamma^{-1} \mathrm{rad\ s}^{-1}$

corresponding to a frequency $\nu_B = \frac{\omega_B}{2\pi} = 2.8 \frac{B}{\mu G} \gamma^{-1} \mathrm{Hz}$

Acceleration

The acceleration $a = v_\perp \omega_B$ is perpendicular to the velocity, v

Let's orient our axes so that v is instantaneously in the x-direction and a is in the y-direction

Then $a_y = v_\perp \omega_B$ and the 4-acceleration is

$$a^\mu = \frac{dU^\mu}{d\tau} = \frac{d}{d\tau}\begin{bmatrix}\gamma c \\ \gamma \boldsymbol{v}\end{bmatrix} = \gamma \frac{d}{dt}\begin{bmatrix}\gamma c \\ \gamma \boldsymbol{v}\end{bmatrix} = \gamma^2 \begin{bmatrix}0 \\ d\boldsymbol{v}/\boldsymbol{dt}\end{bmatrix} = \gamma^2 \begin{bmatrix}0 \\ 0 \\ v_\perp \omega_B \\ 0\end{bmatrix}$$

since γ is constant

In the instantaneous particle rest frame, S', we find $a'^\mu = a^\mu$
(since the *x* and *t* components are both zero)

Hence, the 3-acceleration in the rest-frame has magnitude $a' = a'_y = \gamma^2 v_\perp \omega_B$

Total radiated power

The power radiated in the particle rest frame is given by the Larmor formula

$$P' = \frac{2(ea')^2}{3c^3} = \frac{2\left(e\gamma^{\mathbf{2}}v_\perp\omega_B\right)^2}{3c^3} = \frac{2e^2\gamma^{\mathbf{4}}{v_\perp}^2}{3c^3}\left(\frac{eB}{\gamma mec}\right)^2$$

$$= \frac{2\gamma^{\mathbf{2}}{v_\perp}^2}{3c}\left(\frac{e^2}{m_e c}\right)^2 8\pi\left(\frac{B^2}{8\pi}\right) = \frac{2\gamma^{\mathbf{2}}{v_\perp}^2}{c^2}\sigma_T U_B c$$

What about the power radiated in the lab frame, $P_e = dE/dt$?

We have

$dE = \gamma(dE' + v\mathrm{d}p'_x) = \gamma\, dE'$ because $\mathrm{d}p'_x = 0$ (forward/backward symmetry)
$dt = \gamma(dt' + v\mathrm{d}\mathrm{x}') = \gamma\, dt'$ because the particle is at rest in S'

So $P_e = dE/dt = dE'/dt' = P' = 2\sigma_T U_B \beta_\perp^2\, \gamma^{\mathbf{2}}$

Rate of energy loss

$$P_e = 2c\sigma_T U_B \beta_\perp^2 \gamma^{\mathbf{2}}$$

Averaged over an isotropic distribution of pitch-angles

$$\langle \beta_\perp^2 \rangle = \beta^2 \langle \sin^2 \alpha \rangle = \frac{2}{3}\beta^2 \qquad \Rightarrow \quad \langle P_e \rangle = \frac{4}{3} c\, \sigma_T\, U_B\, \beta^{\,2} \gamma^{\mathbf{2}}$$

As the electrons lose energy, their Lorentz factor decreases slowly according to

$$\langle P_e \rangle = -\frac{d}{dt}(\gamma m_e c^2) \Rightarrow \frac{d\gamma}{dt} = -\frac{\langle P_e \rangle}{m_e c^2} = -\frac{4\, c\, \sigma_T\, U_B\, \beta^{\,2} \gamma^{\mathbf{2}}}{3 m_e c^2}$$

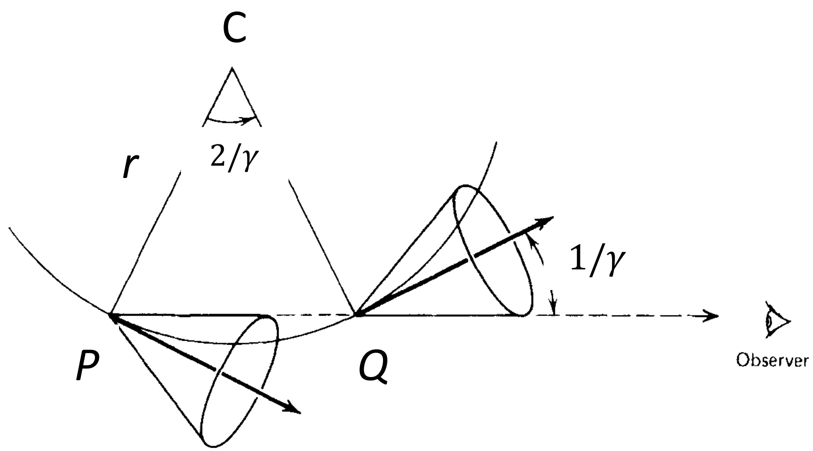
so the energy loss timescale is

$$\tau_{loss} \equiv -\frac{\gamma}{d\gamma/dt} = \frac{3 m_e c}{4c\, \sigma_T\, U_B\, \beta^{\,2}\, \gamma} = \frac{2.5\times 10^{13}\ \mathrm{yr}}{(B/\mu G)^2\, \gamma}$$

For $B = 5\ \mu G$ and $\gamma = 10^4$, we find $\tau_{loss} = 100\ \mathrm{Myr}$

Spectrum of the radiation received

Consider the plane in which the particle is moving instantaneously



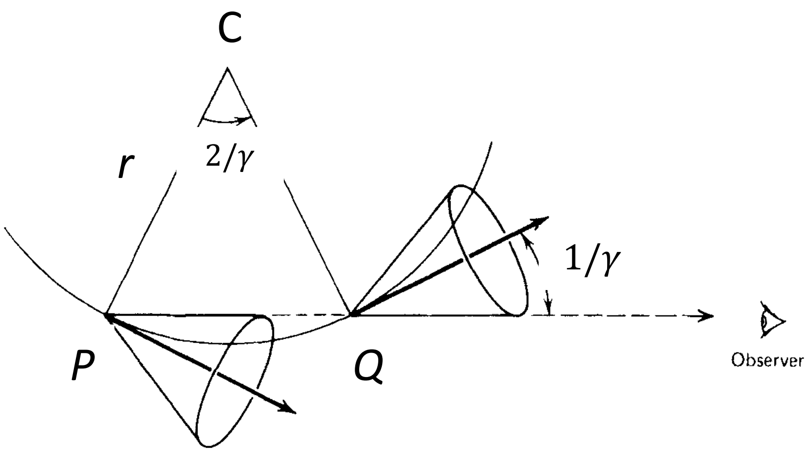
Because of beaming, an observer only sees radiation during a small fraction of the electron orbit, when the electron is between points *P* and *Q*

The angle $\angle PCQ \sim 2/\gamma$

The radius of curvature, *r*, is given by

$$\gamma m_e \frac{v^2}{r} = \left|\frac{q}{c} v \times B\right| = \frac{evB\sin\alpha}{c} \Rightarrow \quad r = \frac{\gamma m_e vc}{eB\sin\alpha} = \frac{v}{\omega_B \sin\alpha}$$

Spectrum of the radiation received



$$r = \frac{v}{\omega_B \sin\alpha}$$

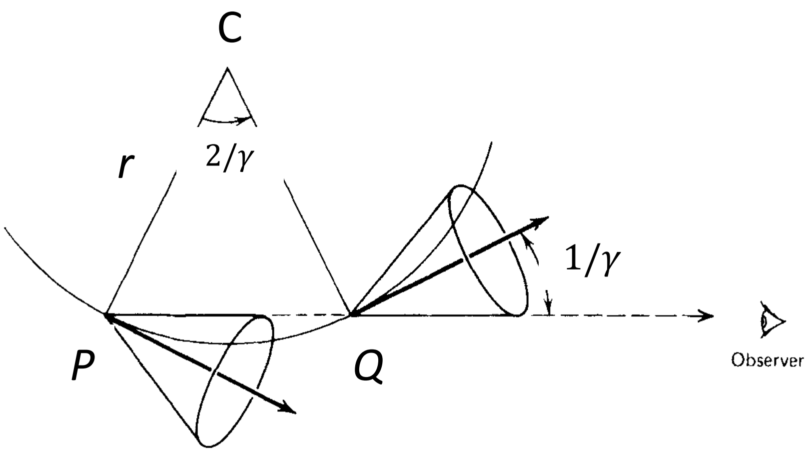
Time taken for particle to travel from P to Q, $\Delta t = \frac{r\,\angle PCQ}{v} = \frac{2r}{\gamma v} = \frac{2}{\gamma \omega_B \sin\alpha}$

This is ***not***, however, the duration of the pulse that is received, because of light travel time effects

Radiation emitted from P at time t_p arrives at time $t_p^A = t_p + l_p/\mathrm{c}$
Radiation emitted from Q at time t_q arrives at time $t_q^A = t_q + l_q/\mathrm{c}$
where l_p and l_q are the distances from the observer

Spectrum of the radiation received

Consider the plane in which the particle is moving instantaneously



$$r = \frac{v}{\omega_B \sin\alpha}$$

Length of the observed pulse of radiation is

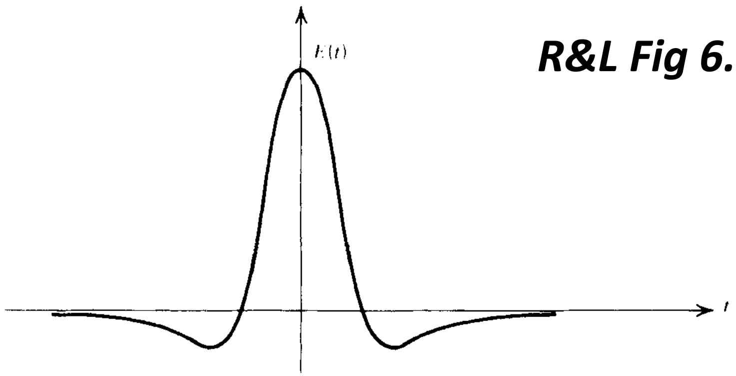
$$\Delta t^A = t_q^A - t_p^A = (t_q - t_p) + (l_q - l_p)/c = 2r/[\gamma v] - 2r\sin(1/\gamma)/c$$

$$= \frac{2}{\gamma\omega_B\sin\alpha} - \frac{2v}{c\omega_B\sin\alpha}\sin(1/\gamma) = \frac{2}{\gamma\omega_B\sin\alpha}(1 - \beta\,\gamma\sin(1/\gamma))$$

$$\sim \frac{2}{\gamma\omega_B\sin\alpha}\left(1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{1}{6\gamma^2}\right)\right) \sim \frac{4}{3\gamma^3\omega_B\sin\alpha} \quad \text{for } \gamma \gg 1$$

Spectrum of the radiation received

The E-field shows a pulse of width $\Delta t^A \sim (\gamma^3 \omega_B \sin\alpha)^{-1}$



R&L Fig 6.3

which implies that the emitted spectrum contains angular frequencies up to $\sim \gamma^3 \omega_B \sin\alpha \;\; = \gamma^2 \frac{qB}{m_e c} \sin\alpha$

Note the factor γ^3: for a highly relativistic electron this can be way higher than the Larmor frequency

Power spectrum of synchrotron radiation

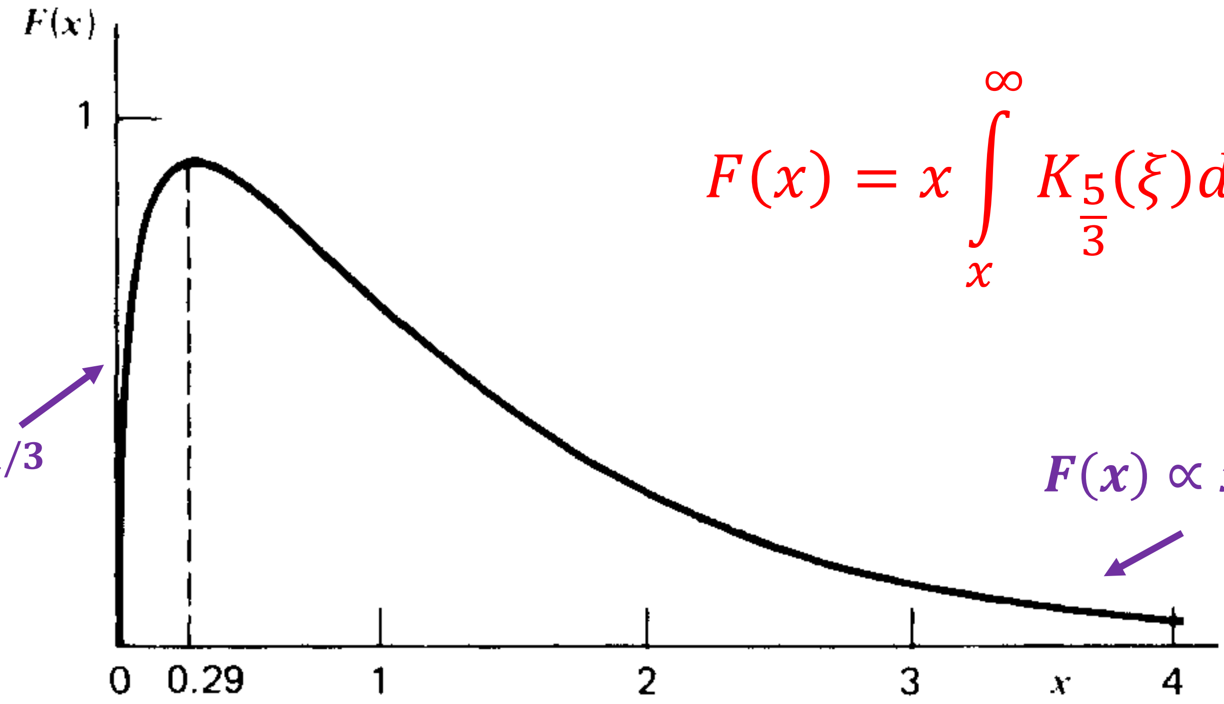
The power spectrum is obtained by taking the Fourier transform of the pulse shape (and squaring its magnitude) – see R&L Section 6.4 for details

If we define the "cutoff" (angular frequency) by $\omega_c = \frac{3}{2}\gamma^3\omega_B \sin\alpha$, the

monochromatic power emitted per electron is $\frac{dP}{d\omega} = \frac{\sqrt{3}}{2\pi}\frac{q^3 B \sin\alpha}{m_e c^2} F\left(\frac{\omega}{\omega_c}\right)$

where

$$F(x) = x\int_x^\infty K_{\frac{5}{3}}(\xi)\,d\xi$$



$F(x) \propto x^{1/3}$

$F(x) \propto x^{1/2}e^{-x}$

R&L Fig 6.6

Lecture 18
Synchrotron radiation II

Goals: understand

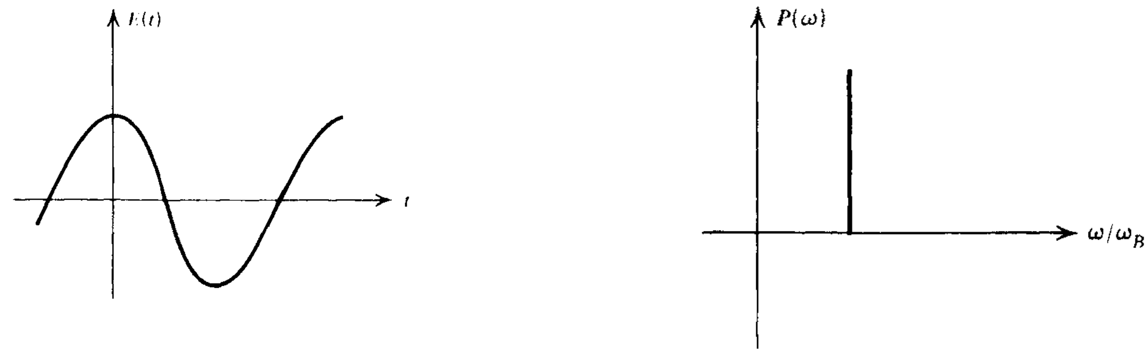
Synchrotoron spectrum for monoenergetic electrons

The synchrotron spectrum expected with a power-law distribution of electron energies

Inverse Compton radiation

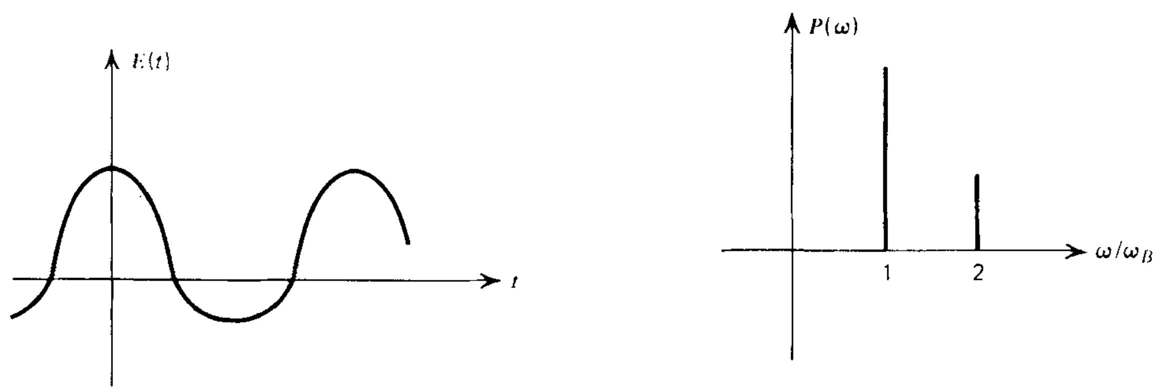
Transition from cyclotron ($\gamma \sim 1$) to synchrotron ($\gamma \gg 1$) radiation

Clearly, in the non-relativistic limit ("cyclotron radiation") there is no beaming and the E-field is just a sine wave ➔ the power spectrum is a delta function at $\omega = \omega_B$



R&L Fig 6.8

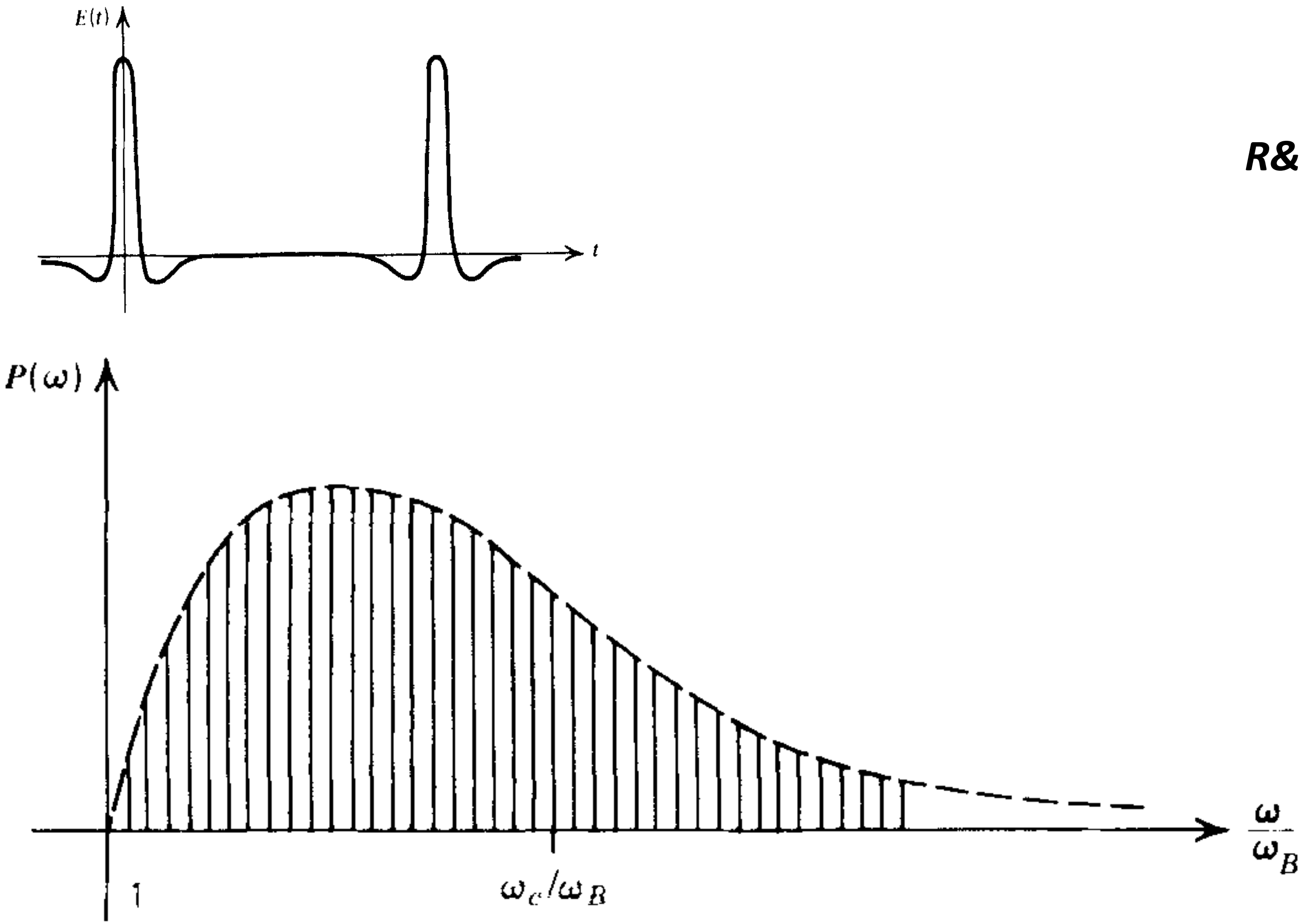
As γ increases, the waveform starts to get distorted by the effects of beaming and we start to get harmonics



R&L Fig 6.9

Transition from cyclotron ($\gamma \sim 1$) to synchrotron ($\gamma \gg 1$) radiation

As γ becomes very large, we see higher and higher harmonics which eventually "wash-out" to yield a continuum



R&L Fig 6.10

Power spectrum for synchrotron ($\gamma \gg 1$) radiation

Once the spectrum is well approximated by a continuum, the spectrum is

$$\frac{dP}{d\omega} = \frac{\sqrt{3}}{2\pi}\frac{q^3 B \sin\alpha}{m_e c^2} F\left(\frac{\omega}{\omega_c}\right)$$

Key features of this result:

1) The total power is

$$P = \int_0^\infty \frac{dP}{d\omega} d\omega = \frac{\sqrt{3}}{2\pi}\frac{q^3 B\, \omega_c \sin\alpha}{m_e c^2}\int_0^\infty F(x)dx$$

$$= \frac{\sqrt{3}}{2\pi}\frac{q^3 B \sin\alpha}{m_e c^2}\frac{3}{2}\gamma^3 \omega_B \sin\alpha \frac{8\pi}{9\sqrt{3}} = \frac{4}{9}\frac{q^3 B \sin\alpha}{m_e c^2}\frac{3}{2}\gamma^3 \frac{qB}{m_e \gamma c}\sin\alpha$$

$$= \frac{2}{3}\frac{q^4 B^2}{m_e{}^2 c^3}\gamma^2 \sin^2\alpha \sim 2c\sigma_T U_B \beta_\perp^2 \gamma^2$$

(since $\beta_\perp \sim \sin\alpha$). This agrees with our previous result from the Larmor formula.

Power spectrum for synchrotron ($\gamma \gg 1$) radiation

Key features of $\frac{dP}{d\omega} = \frac{\sqrt{3}}{2\pi}\frac{q^3 B \sin\alpha}{m_e c^2} F\left(\frac{\omega}{\omega_c}\right)$

2) The function *F* only depends on ω/ω_c and is independent of ω/ω_B

Explanation: the relativistic beaming factor is a function of $\gamma\theta$

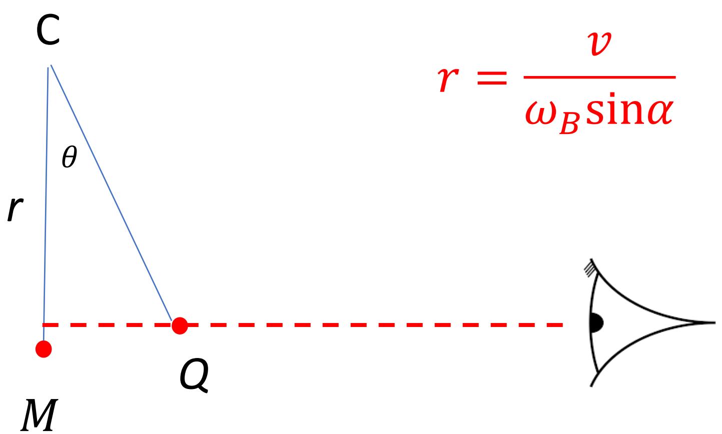
Recall that $\frac{dP_r}{d\Omega} = \frac{16\gamma^4}{(1+\gamma^2\theta^2)^4}\frac{P\prime}{4\pi} = \frac{1}{(1+\gamma^2\theta^2)^4}\left(\frac{dP_r}{d\Omega}\right)_{max}$

So $E(t)/E_{max}$ is a universal function of $\gamma\theta$

We now need to return to an earlier figure to determine how the arrival time is related to θ

Spectrum of the radiation received

We will now call M the midpoint of the region where the emitted radiation is beamed towards us



$$r = \frac{v}{\omega_B \sin\alpha}$$

The arrival time of radiation emitted from point Q relative to the middle of the pulse is

$$\Delta t^A = t_q^A - t_m^A = (t_q - t_m) + (l_q - l_m)/c = r\theta/v - r\sin\theta/c$$

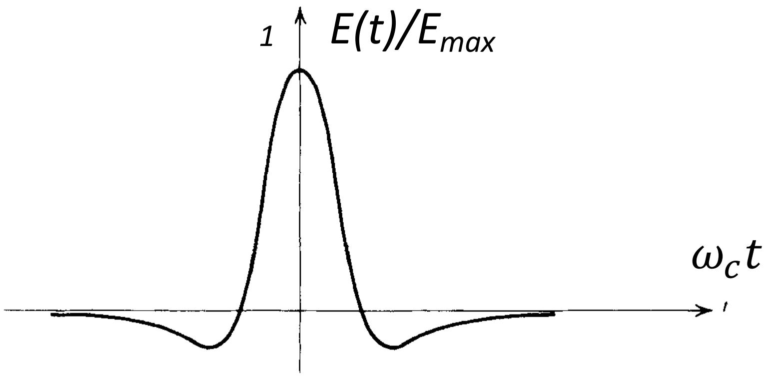
$$= \frac{\theta}{\omega_B \sin\alpha} - \frac{v}{c\omega_B \sin\alpha}\sin\theta = \frac{\theta}{\omega_B \sin\alpha} - \frac{1}{\omega_B \sin\alpha}\left(1 - \frac{1}{2\gamma^2}\right)\left(\theta - \frac{\theta^3}{6}\right)$$

$$\sim \frac{1}{\omega_B \sin\alpha}\left(\frac{\theta}{2\gamma^2} + \frac{\theta^3}{6}\right) = \frac{1}{\gamma^3 \omega_B \sin\alpha}\left(\theta\gamma - \frac{\theta^3\gamma^3}{6}\right) = \frac{3}{2\omega_c}\left(\theta\gamma - \frac{\theta^3\gamma^3}{6}\right)$$

So E and $\omega_c t$ are both functions of $\theta\gamma$ alone ➔ E is a fixed function of $\omega_c t$

The pulse shape is a fixed function of $\omega_c t$

As plotted below (*E(t)/E_{max}* versus $\omega_c t$), the pulse shape looks the same whatever γ and *B*



i.e. $E(t)/E_{\max} = g(\omega_c t)$

So when we take the Fourier transform, we get

$$\tilde{E}_T(\omega) = \frac{1}{2\pi}\int e^{i\omega t}\, E(t)dt = \frac{1}{2\pi}\int e^{i\omega t}\, g(\omega_c t)dt = \frac{1}{2\pi\omega_c}\int e^{i(\omega/\omega_c)\xi}\, g(\xi)d\xi$$

The integral is a function of ω/ω_c alone, and thus the spectral shape (i.e. $P(\omega)/Pmax$ is a function of ω/ω_c)

Meaning: spectrum is identical for ($B = 1\mu G, \gamma = 2\times 10^4$) and ($B = 4\mu G, \gamma = 10^4$)
(Both cases have the same ω_c which is $\propto\ \gamma^3\omega_B \propto\ \gamma^2 B$

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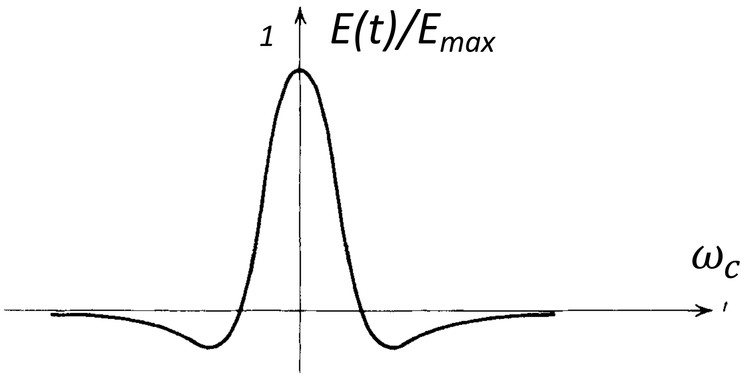
As plotted below (*E(t)/E_{max}* versus $\omega_c t$), the pulse shape looks the same whatever γ and *B*



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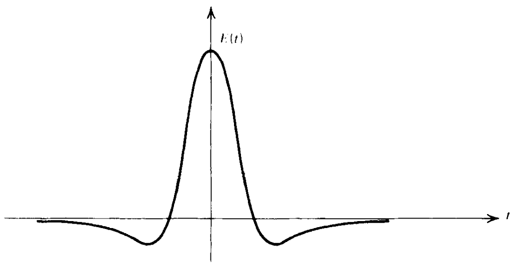
(Both cases have the same ω_c which is $\propto\ \gamma^3\omega_B \propto\ \gamma^2 B$

Power spectrum for synchrotron ($\gamma \gg 1$) radiation

Key features of $\frac{dP}{d\omega} = \frac{\sqrt{3}}{2\pi}\frac{q^3 B \sin\alpha}{m_e c^2} F\left(\frac{\omega}{\omega_c}\right)$

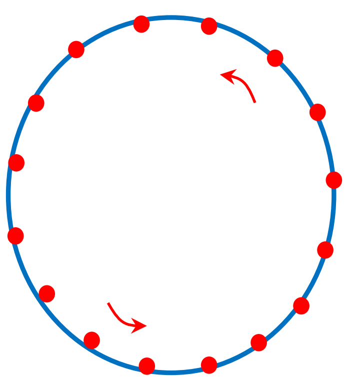


3) The function $F \to 0$ as $\omega \to 0$

This must mean that $\tilde{E}_T(\omega) = \frac{1}{2\pi}\int e^{i\omega t} E(t)dt \to 0$ as $\omega \to 0$

which implies that the pulse has zero net area: $\int E(t)dt = 0$

This makes sense, because integrating over one orbit is equivalent to placing orbiting electrons everywhere on a circle



This is just a current loop, which has no electric dipole moment (and a constant magnetic dipole moment) ➔ no long range *E*-field decreasing as as only *1/R* ➔ no radiation

Synchrotron spectrum for a power-law distribution of energies

So far, we have only considered the emission of relativistic electrons with a single energy $\gamma m_e c^2$

$$\frac{dP}{d\omega} = \frac{\sqrt{3}}{2\pi}\frac{q^3 B \sin\alpha}{m_e c^2} F\left(\frac{\omega}{\omega_c}\right)$$

where ω_c is proportional to $\gamma^2 B$

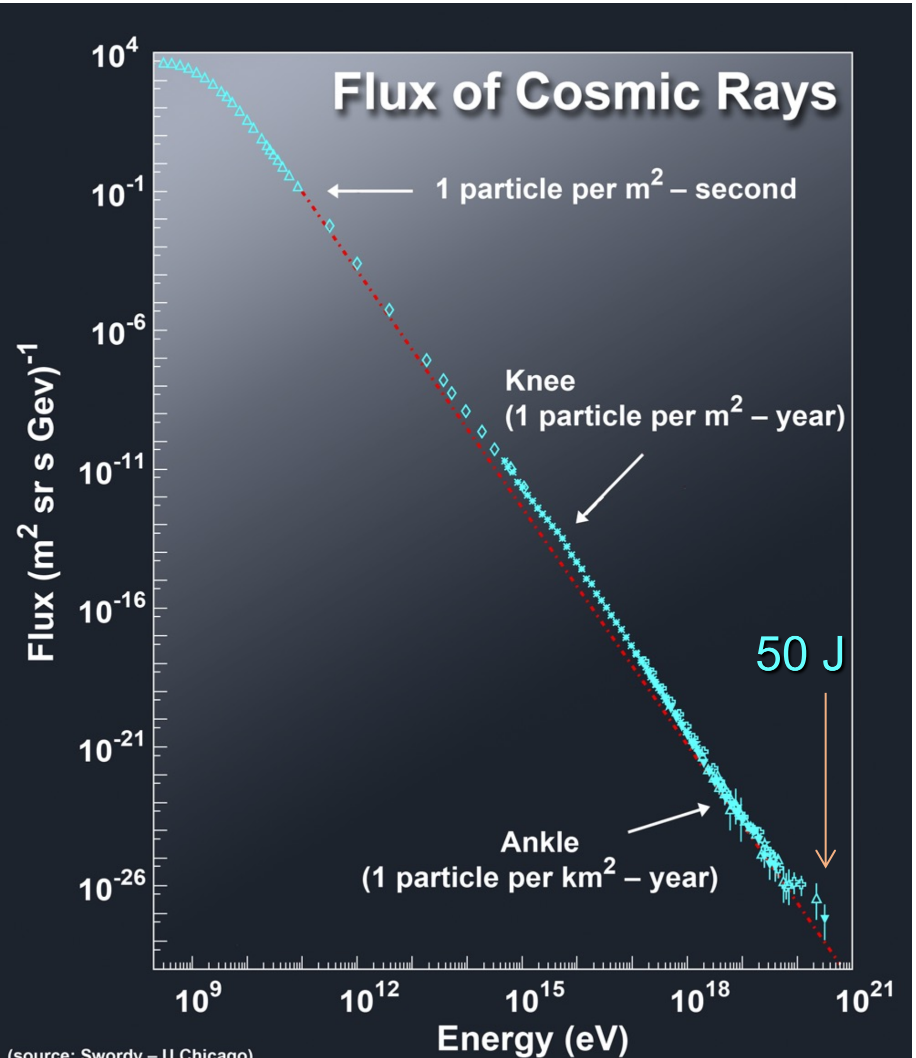
But cosmic rays have a roughly power law distribution of energies over a huge range of γ

Let's suppose that the number of electrons with Lorentz factor γ to $\gamma + d\gamma$ is $dN = \gamma^{-p}\, d\gamma$ where p is some positive number.

The integral $\int dN$ will diverge either at large γ (if $p < 1$) or small γ (if $p > 1$) (or both if $p = 1$), so let's assume that this power law applies only over some large range of Lorentz factors, γ_1 to γ_2

Cosmic ray energy spectrum

CR are observed over a remarkable range of energies



Cosmic ray spectrum (credit: HAP / A. Chantelauze)

Maximum energy detected to date: 50 J

(~ K.E. of a fastball in Little League baseball)

Total energy density ~ 1 eV cm^{-3}

… somewhat LARGER than that of

starlight, the CMB, or the Galactic B-field

Table 1.5 Energy Densities in the Local ISM

Component	$u(\mathrm{eV\,cm^{-3}})$	Note
Cosmic microwave background ($T_{\mathrm{CMB}} = 2.725$ K)	0.265	*a*
Far-infrared radiation from dust	0.31	*b*
Starlight ($h\nu < 13.6$eV)	0.54	*c*
Thermal kinetic energy $(3/2)nkT$	0.49	*d*
Turbulent kinetic energy $(1/2)\rho v^2$	0.22	*e*
Magnetic energy $B^2/8\pi$	0.89	*f*
Cosmic rays	1.39	*g*

a Fixsen & Mather (2002).
b Chapter 12.
c Chapter 12.
d For $nT = 3800\,\mathrm{cm^{-3}\,K}$ (see §17.7).
e For $n_{\mathrm{H}} = 30\,\mathrm{cm^{-3}}$, $v = 1\,\mathrm{km\,s^{-1}}$, or $\langle n_{\mathrm{H}}\rangle = 1\,\mathrm{cm^{-3}}$, $\langle v^2\rangle^{1/2} = 5.5\,\mathrm{km\,s^{-1}}$.
f For median $B_{\mathrm{tot}} \approx 6.0\,\mu$G (Heiles & Crutcher 2005).
g For cosmic ray spectrum X3 in Fig. 13.5.

Draine, 2011

Synchrotron spectrum for a power-law distribution of energies

The power radiated by this collection of electrons is then

$$\frac{dP_{\text{tot}}}{d\omega} = \int \frac{dP}{d\omega} dN = \int_{\gamma_1}^{\gamma_2} \frac{dP}{d\omega} \gamma^{-p} d\gamma \propto \int_{\gamma_1}^{\gamma_2} F(\omega/\omega_c)\ \gamma^{-p} d\gamma$$

Now let's define $x \equiv \omega/\omega_c = k\omega/\gamma^2 \quad \Rightarrow \gamma = \sqrt{k\omega/x}$

where k is some constant that depends on *B* but not γ

$$\frac{dP_{\text{tot}}}{d\omega} \propto \int_{x_1}^{x_2} F(x)\ (k\omega/x)^{-p/2}\ d(\sqrt{k\omega/x}) \propto \omega^{(1-p)/2} \int_{x_1}^{x_2} F(x)\, x^{(p-3)/2} dx$$

If ω_c for $\gamma_1 \ll\ \omega \ll\ \omega_c$ for γ_2, then $x_1 \ll 1$ and $x_2 \gg 1$. We can then take the integral from 0 to ∞, and the only frequency dependence is $\omega^{(1-p)/2}$

Synchrotron spectrum for a power-law distribution of energies

Through this analysis, we find that a power law distribution of electron energies yields a power law distribution of emitted frequency but the power-law indices are different

$$dN = \gamma^{-p}\, d\gamma \qquad \Rightarrow \qquad F_{\upsilon} \propto \upsilon^{-s}$$

where $s = \frac{1}{2}(p-1)$

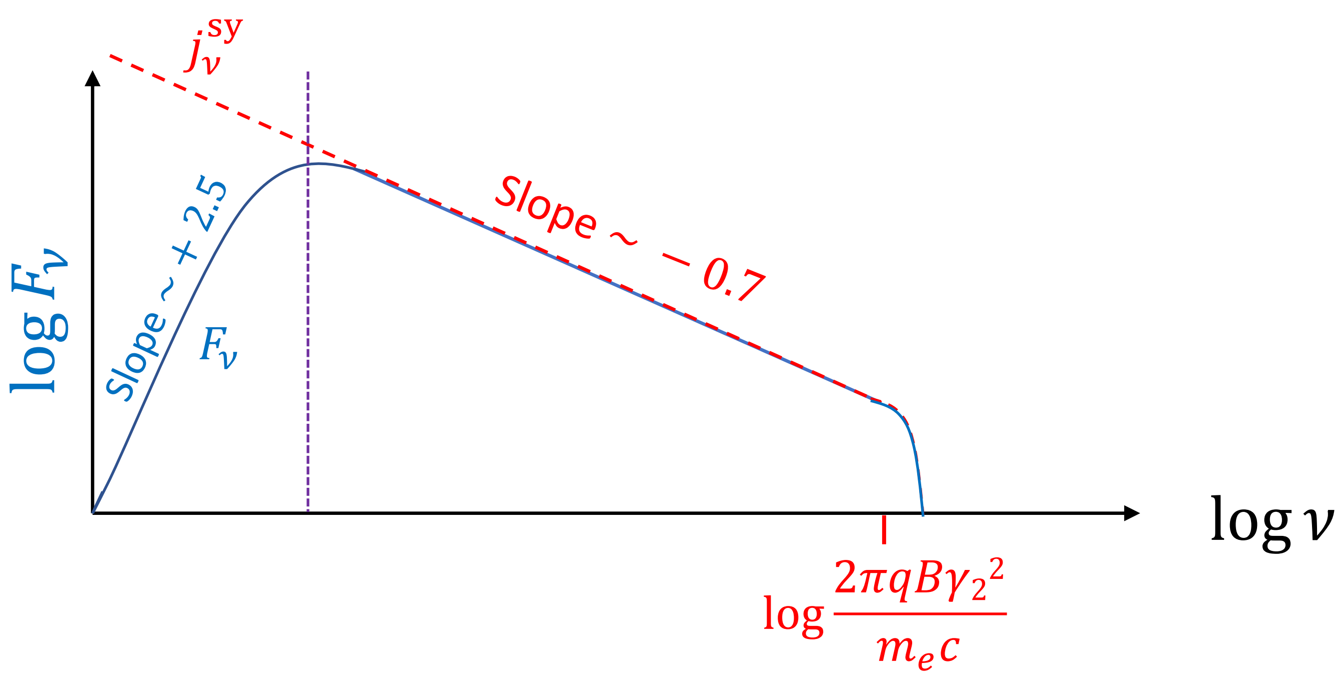
We call s the "spectral index"
(equals –2 in the Rayleigh Jeans limit, ~0.1 for Bremsstrahlung)

Typically, the spectral index for synchrotron is ~0.7, corresponding to $p = 2s + 1 \sim 2.4$

Synchrotron self-absorption

As in the case of Bremsstrahlung, synchrotron radiation can become optically-thick at low frequencies

However, with a power-law distribution of energies instead of a thermal distribution, the spectral index is –5/2 instead of -2.



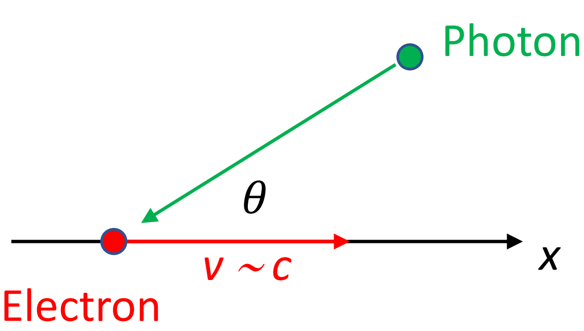
Inverse Compton radiation

In addition to emitting synchrotron radiation, relativistic electrons also scatter radiation, especially the cosmic microwave background (CMB). This gives rise to to what is known as “inverse Compton radiation”

CMB photons have a typical energy

$$\epsilon \sim kT_{\mathrm{CMB}} = \ 3.77 \times 10^{-16}\mathrm{erg} \ \sim 2.36 \times 10^{-4}\ \mathrm{eV}$$

Consider a photon incident at angle θ to the particle velocity as shown below



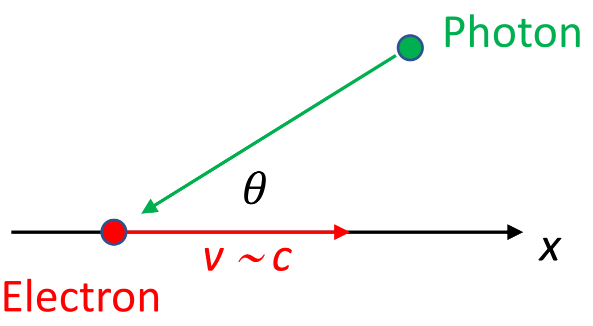
Picture in the lab frame S
(observer at rest in Galaxy)

Inverse Compton radiation

In the rest frame of the electron, S', the photon energy is $\epsilon' = \gamma\epsilon(1+\beta\mu)$

For an isotropic distribution of photons in the lab frame, the average value of $(1+\beta\cos\theta)$ is 1, and thus the typical CMB photon is *highly blueshifted* as viewed in the electron rest frame: $\langle\epsilon'\rangle = \gamma\epsilon$

Q1: γ is the *average* value of ϵ'/ϵ, but what is the *minimum* value (if $\gamma \gg 1$)?



Inverse Compton radiation

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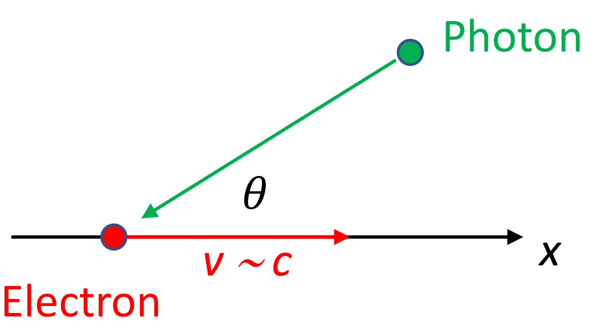
Q1: γ is the *average* value of ϵ'/ϵ, but what is the *minimum* value (if $\gamma \gg 1$)?

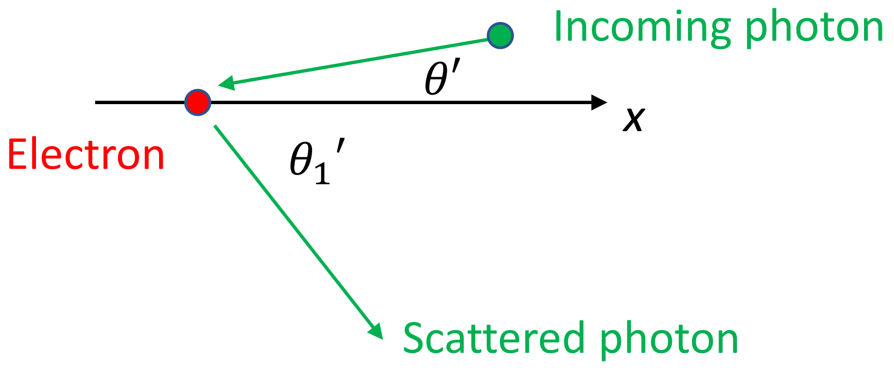
Answer: at $\mu = -1$ (i.e. $\theta = \pi$ $\Rightarrow$ photon coming directly from behind)
we have $\epsilon' = \gamma\epsilon(1 - \beta) \sim \gamma\epsilon/[2\gamma^2] = \epsilon/[2\gamma]$
Some photons (with θ ***very close*** to π) are redshifted, but *most* are blueshifted



Inverse Compton radiation

In the rest frame of the electron, S', the scattering is coherent provided $\epsilon' \ll m_e c^2$ (recoil negligible)

Thus, the energy of the scattered photon in the electron rest frame is
$\epsilon_1{}' = \epsilon' = \gamma\epsilon(1 + \beta\cos\theta)$



Picture in the electron rest frame S'

Suppose the scattered photon emerges at angle $\theta_1{}'$ to the x-axis (as measured in the electron rest frame S')

Converting back to the lab frame, this scattered photon has energy
$\epsilon_1 = \gamma\epsilon_1{}'(1 + \beta\cos\theta_1{}')$

Inverse Compton radiation

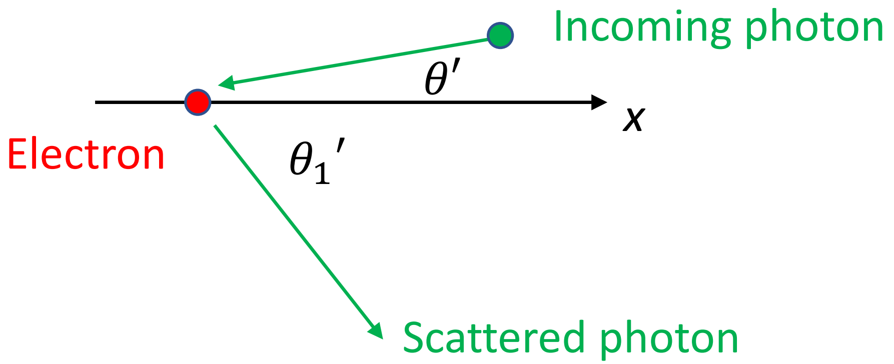
So we have

$\epsilon \sim kT_{\mathrm{CMB}}$ Energy of CMB photon in S frame

$\epsilon' = \gamma\epsilon(1+\beta\cos\theta)$ Energy of CMB photon in the S' frame

$\epsilon_1' = \epsilon'$ Energy of scattered photon in the S' frame

$\epsilon_1 = \gamma{\epsilon_1}'(1+\beta\cos{\theta_1}')$ Energy of scattered photon in the S frame



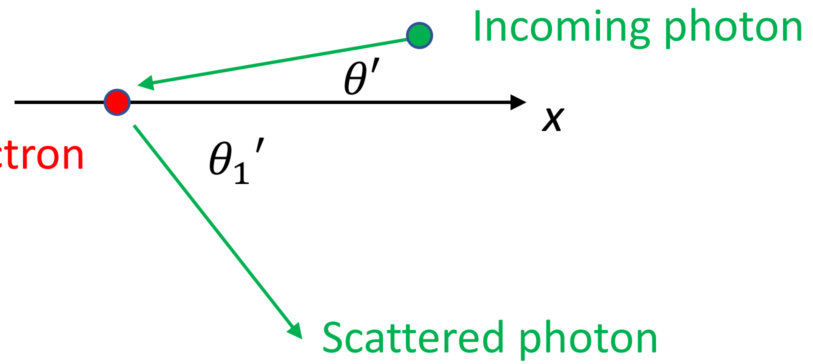
Picture in the electron rest frame S

Q2: what is the typical energy of the scattered photon in the lab frame?

Inverse Compton radiation

So we have

$\epsilon \sim kT_{\mathrm{CMB}}$	Energy of CMB photon in S frame
$\epsilon' = \gamma\epsilon(1+\beta\cos\theta)$	Energy of CMB photon in the S' frame
$\epsilon_1' = \epsilon'$	Energy of scattered photon in the S' frame
$\epsilon_1 = \gamma\epsilon_1'(1+\beta\cos\theta_1')$	Energy of scattered photon in the S' frame



Picture in the electron rest frame S

Q2: what is the typical energy of the scattered photon in the lab frame?

Answer: $\epsilon_1 = \gamma^2\epsilon\,(1+\beta\cos\theta)(1+\beta\cos\theta_1') = \gamma^2\epsilon$

Average = 1 because CMB is isotropic in S

Average = 1 because of forward-backward symmetry in S'

Inverse Compton radiation

$$\epsilon_1 = \gamma^2 \epsilon \, (1 + \beta \cos\theta)(1 + \beta \cos{\theta_1}')$$

So, the lab frame, the scattered radiation has typical energy $\gamma^2\epsilon$ and maximum energy $4\,\gamma^2\epsilon$

For a Lorentz factor of 10^4, this yields a typical energy

$$\epsilon_1 \sim 10^8\,(2 \times 10^{-4}\ \mathrm{eV}) = 20\ \mathrm{keV}$$

(microwaves scattered to yield X-rays!)

This process works so long as ϵ' is less than $\sim m_e c^2$,
i.e. for γ up to $\sim m_e c^2/\,\epsilon = \dfrac{511\ \mathrm{keV}}{2 \times 10^{-4}\ \mathrm{eV}} \sim 2.5 \times 10^9$

In principle, it can therefore produce gamma rays with energies up to
$\epsilon_1 \sim \gamma^2\epsilon \sim (m_e c^2/\epsilon)^2\,\epsilon \sim (m_e c^2)^2/\epsilon \sim 10^{15}$ eV

Lecture 19
Inverse Compton radiation II

Goals: understand

Spectrum of IC radiation:

monoenergetic case

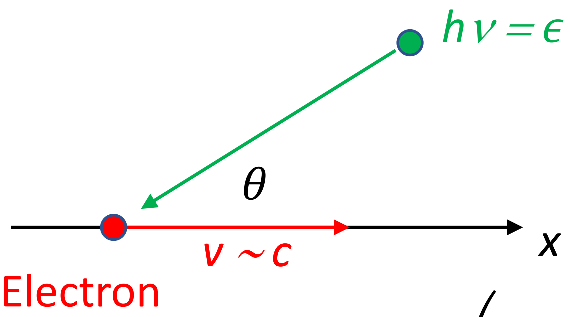
power-law distribution

Compton scattering by non-relativistic thermal electrons

Inverse Compton radiation

Let's work out the power generated by this process. To do so, we need to find the mean intensity of the radiation in the electron rest frame, $J' = cu'/4\pi$.
The scattered power will then be $P' = 4\pi J'\sigma_T$

Consider first a beam of photons travelling at angle θ to the direction of motion



The photon 4-momentum is $p^\mu = \begin{pmatrix} \epsilon/c \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \epsilon/c \\ -\epsilon\cos\theta/c \\ -\epsilon\sin\theta/c \\ 0 \end{pmatrix}$

Suppose there are n photons per unit volume. Then $u = n\epsilon$ and $u' = n'\epsilon'$

Inverse Compton radiation

How does n transform to the S' frame? We can construct a 4-vector exactly analogous to the 4-current $j^{\mu} = q\binom{nc}{n\boldsymbol{v}}$. It is $V^{\mu} = \binom{nc}{n\boldsymbol{v_p}} = \begin{pmatrix} nc \\ -nc\cos\theta \\ -nc\sin\theta \\ 0 \end{pmatrix}$

So n must transform exactly the same way as ϵ

In other words, the photon density in the S' frame is $n' = \gamma n(1 + \beta\cos\theta)$ exactly in the same way that $\epsilon' = \gamma\epsilon(1 + \beta\cos\theta)$

In the S' frame, the energy density associated with these photons is

$$u' = n'\epsilon' = \gamma^2(1 + \beta\cos\theta)^2\, n\,\epsilon = \gamma^2(1 + \beta\cos\theta)^2\, u$$

If we now consider an isotropic distribution of photons instead of a beam, we obtain $u' = \gamma^2 \langle 1 + 2\beta\cos\theta + \beta^2\cos^2\theta \rangle\, u = \gamma^2\left(1 + \frac{\beta^2}{3}\right)u$

Inverse Compton radiation

The scattered power is then $P' = 4\pi J' \sigma_T = cu'\sigma_T$

where $J' = cu'/\,4\pi$ is the mean intensity of the radiation in the S' frame

As we discussed previously, the scattered power is invariant, so in the lab frame we also have

$$P_s = P' = cu'\sigma_T = \gamma^2\left(1 + \frac{\beta^2}{3}\right)u\sigma_T c$$

This is the rate at which power is added to the radiation field due to blueshifted scattered photons. We have to subtract the rate at which CMB photons are removed, $u\sigma_T c$, to obtain the net power produced by the inverse Compton process

$$P_{IC} = P' = cu'\sigma_T - u\sigma_T c = \left(\gamma^2 + \frac{\beta^2\gamma^2}{3} - 1\right)u\,\sigma_T c = \frac{4}{3}\beta^2\gamma^2 u\,\sigma_T c$$

using $\gamma^2 - 1 = \beta^2\,\gamma^2$

This result is true even if γ isn't >> 1

Inverse Compton radiation

$P_{IC} = \frac{4}{3}\beta^2\gamma^2 u\ \sigma_T c$ may look vaguely familiar!

Recall the result we got for synchrotron radiation in Lecture 17

$P_{\rm synch} = \frac{4}{3} c\ \sigma_T\ U_B\ \beta^{\,2}\ \gamma^{\mathbf{2}}$

The ratio of these is just the ratio of the photon energy density to the magnetic field energy density

$$\frac{P_{\rm IC}}{P_{\rm synch}} = \frac{u}{U_B} = \frac{aT_{\rm CMB}{}^4}{B^2/8\pi}$$

Putting in numbers, we find (amazingly) that for $T_{\rm CMB}$ = 2.73 K

$$\frac{P_{\rm IC}}{P_{\rm synch}} = \left(\frac{3.25\ \mu G}{B}\right)^2 \sim 0.3$$

Spectrum of Inverse Compton Radiation for a single value of γ

In the limit of large γ, the spectrum extends to $\epsilon_{1,\max} = 4\,\gamma^2\epsilon$, and can be expressed as a function of $x = \epsilon_1/\epsilon_{1,\max}$ alone

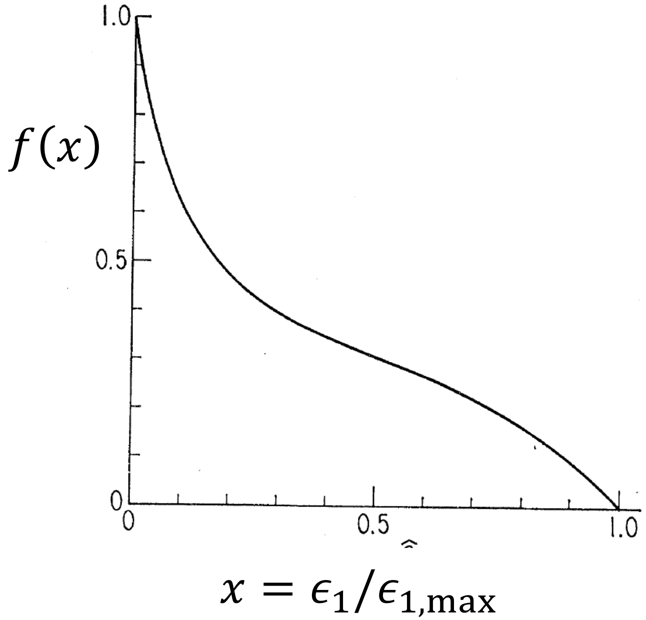
i.e. $\frac{dP}{d\epsilon_1} = \left(\frac{dP}{d\epsilon_1}\right)_{\max} f(x)$

This makes sense, because $\epsilon_1 = \gamma^2\epsilon\,(1+\beta\cos\theta)(1+\beta\cos\theta_1{}')$
➔ $x = \epsilon_1\,/\epsilon_{1,\max}$ =$(1+\beta\cos\theta)(1+\beta\cos\theta_1{}')/4$

$f(x)$ can be written in a fairly simple analytic form

$f(x) = 2x\ \ln x + x + 1 - 2x^2$

(Blumenthal & Gould, Rev Mod Physics, 1970)



Spectrum of Inverse Compton Radiation for a power law distribution of γ

As before, we assume a power-law distribution of electron energies $dN = \gamma^{-p}\, d\gamma$ over some wide range of Lorentz factors from γ_1 to γ_2

The power radiated by this collection of electrons is then

$$\frac{dP_{\mathrm{tot}}}{d\epsilon_1} = \int \frac{dP}{d\epsilon_1} dN \propto \int_{\gamma_1}^{\gamma_2} f(\epsilon_1/\epsilon_{1,\mathrm{max}})\ \gamma^{-p} d\gamma$$

Now let's define $x \equiv \epsilon_1/\epsilon_{1,\mathrm{max}} = \epsilon_1/(4\epsilon\gamma^2) \ \Rightarrow \gamma = \sqrt{\epsilon_1/(4\epsilon x)}$

$$\frac{dP_{\mathrm{tot}}}{d\epsilon_1} \propto \int_{x_1}^{x_2} f(x)\,(\epsilon_1/4\epsilon x)^{-p/2}\, d(\sqrt{\epsilon_1/4\epsilon x}) \propto {\epsilon_1}^{(1-p)/2} \int_{x_1}^{x_2} f(x)\, x^{(p-3)/2} dx$$

If $\epsilon_{1,\mathrm{max}}$ for $\gamma_1 \ll \epsilon_1 \ll \epsilon_{1,\mathrm{max}}$ for γ_2, then $x_1 \ll 1$ and $x_2 \gg 1$. We can then take the integral from 0 to ∞, and the only frequency dependence is ${\epsilon_1}^{(1-p)/2}$

Spectral index $s = (p-1)/2$ *exactly as for synchrotron radiation!*

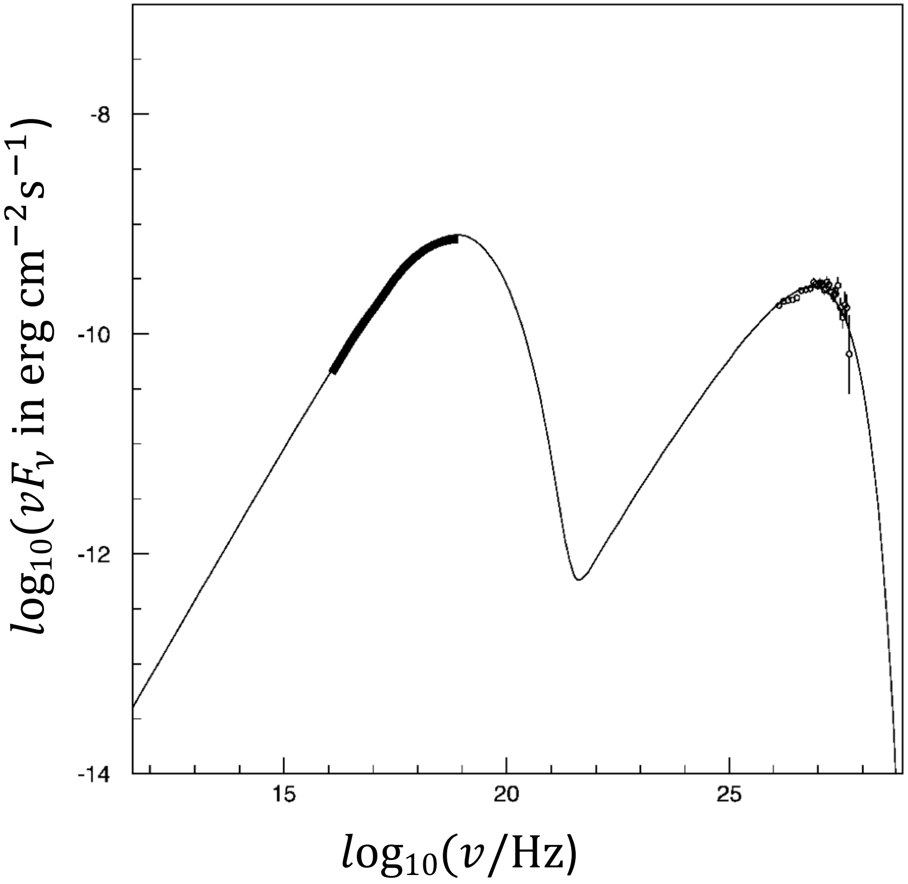
Synchrotron self-Compton process

Not only can relativistic electrons scatter the CMB, they can also scatter the synchrotron radiation that they themselves emit. This is known as the synchrotron self-Compton (SSC) process

The figure at the right shows the spectrum of the blazar Mrk 501 (from Konopelko et al. 2003, ApJ)

The peak around 10^{19} Hz (40 keV) is due to synchrotron radiation

The peak around 10^{27} Hz (4 TeV) is due to SSC

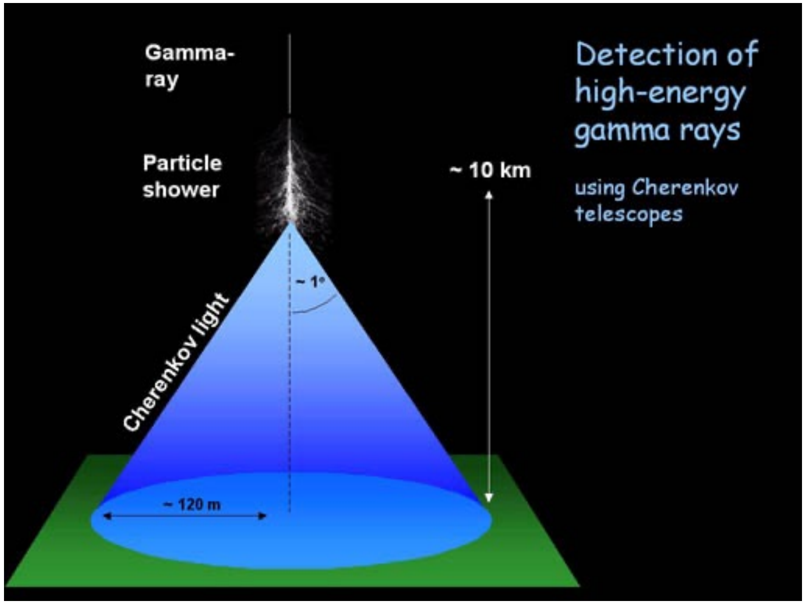


Aside: how are these TeV gamma-rays detected?

While lower energy gamma rays can be observed with satellite observatories, TeV gamma rays can be detected indirectly from the ground

They interact with nuclei in the atmosphere (at an altitude of 10 – 20 km) to produce highly relativistic electron-positron pairs. These are travelling faster than the speed of light in air and give rise to Cherenkov radiation that can be detected from the ground ➔ Cherenkov light pool of diameter ~ 250 m containing ~ 100 photons per m^2 in a pulse of duration ~ few ns.

H.E.S.S. II telescope in Namibia



Scattering in hot (but non-relativistic) thermal plasmas

We now turn to another case that can be treated with our expression

$$P_{IC} = \frac{4}{3}\beta^2\gamma^2 u\, \sigma_T c$$

Clusters of galaxies are typically filled with thermal electrons that are hot but non-relativistic. But our expression above applies even for non-relativistic electrons with $\gamma \sim 1$

Suppose we have n photons of energy ϵ per unit volume. The energy density is $u = n\epsilon$

The rate at which scattering occurs is $S = n\, \sigma_T c$ per electron

The mean energy imparted to each photon on a single scattering is therefore

$$\Delta\epsilon = \epsilon_1 - \epsilon = \frac{P_{IC}}{S} = \frac{\frac{4}{3}\beta^2\gamma^2 u\, \sigma_T\, c}{n\, \sigma_T c} = \frac{4}{3}\beta^2\gamma^2\epsilon \sim \frac{4}{3}\beta^2\epsilon$$

Scattering in hot (but non-relativistic) thermal plasmas

The increase in photon energy per scattering is $\Delta\epsilon = \frac{4\beta^2\epsilon}{3}$, which implies a fractional increase

$$\Delta\ln\epsilon = \frac{\Delta\epsilon}{\epsilon} = \frac{4\beta^2}{3} \text{ (} \ll 1 \text{ for a nonrelativistic electron)}$$

If there is a distribution of electron velocities (e.g. in a thermal plasma) then the average fractional increase per scattering is

$$\langle\Delta\ln\epsilon\rangle = \frac{4\langle\beta^2\rangle}{3}$$

For a Maxwell-Boltzmann distribution of electron densities at temperature T, the mean energy is $\frac{1}{2}m_e\langle v^2\rangle = \frac{1}{2}m_e c^2\langle\beta^2\rangle = \frac{3}{2}kT$ implying

$$\langle\beta^2\rangle = \frac{3kT}{m_e c^2} \text{ and thus } \langle\Delta\ln\epsilon\rangle = \frac{4kT}{m_e c^2}$$

Scattering in hot (but non-relativistic) thermal plasmas

For a single scattering, we had $\langle \Delta \ln \epsilon \rangle = \frac{4kT}{m_e c^2}$

So if a photon suffers N scatterings within a hot plasma, the total average increase in $\ln\epsilon$ is $\frac{4kT}{m_e c^2} N$

This is called the Compton y-parameter: the mean energy of a photon after N scatterings is therefore $\epsilon \exp(y)$

Note that for large N, y does not need to be < 1 even though $\frac{4kT}{m_e c^2} \ll 1$

In clusters of galaxies, CMB photons can be upscattered in energy by this process, which is called the Sunyaev-Zeldovich effect.

Lecture 20

SZ-effect, Propagation of EM waves through a plasma

Goals: understand

Sunyaev-Zeldovich effect

Currents driven when EM waves propagate through a conductive medium

The dispersion relation for a plasma and its astrophysical applications

Faraday rotation

Number of scatterings

If the optical depth is small, $\tau_{es} = \int n_e \sigma_T ds \ll 1$, then almost all photons suffer 0 or 1 scattering. The mean number of scatterings (expectation value) is simply τ_{es}, and the Compton y-parameter is

$$y = \int n_e \sigma_T \frac{4kT}{m_e c^2} ds$$

On the other hand, if $\tau_{es} \gg 1$, then photons suffer of order ${\tau_{es}}^2$ scatterings because they do a random walk.

Recall, each step in the random walk has length $l = 1/(n_e \sigma_T)$, and after N steps the distance travelled is $N^{1/2} l$. So to traverse a cloud of size $\sim D$ the number of required scatterings is given by

$$N^{1/2} l \sim D \Rightarrow N \sim D^2/l^2 = {\tau_{es}}^2$$

So a useful rough estimate is $y \sim \frac{4kT}{m_e c^2} \max(\tau_{es}, {\tau_{es}}^2)$

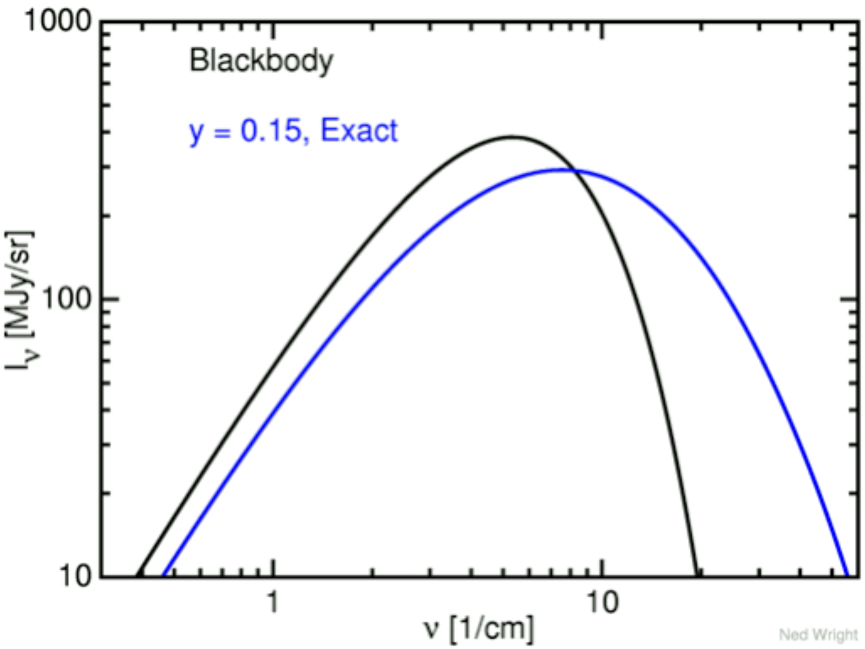
For a rich galaxy cluster, we might have $\tau_{es} \sim 10^{-3}, T \sim 10^8 \mathrm{K} \Rightarrow y \sim 10^{-4}$

The Sunyaev-Zeldovich effect

When CMB photons encounter a galaxy cluster filled with hot gas, their energies are increased by a typical factor e^y

This increases the energy density u *without adding any photons,* so the resultant spectrum is clearly no longer a blackbody.

Proof: the peak of the spectrum shifts to a higher frequency $\nu_{\rm peak}{}' = e^y \nu_{\rm peak}$ and the energy density shifts by the same factor to $u' = e^y u$. But for a blackbody $\nu_{\rm peak} \propto T$ whereas $u \propto T^4$



http://www.astro.ucla.edu/~wright/SZ-spectrum.html

The Sunyaev-Zeldovich effect

To first order, the change in intensity is found to be governed by

$$\frac{\partial I_\nu}{\partial y} = \frac{2h\nu^3}{c^2}\frac{xe^x}{(e^x-1)^2}\left(x\frac{e^x+1}{e^x-1}-4\right) \quad \text{with} \quad x = h\nu/kT_o$$

The lower frequency region is most typically observed, so the SZE shows up as a *decrement* in the intensity

A&A 596, A101 (2016)

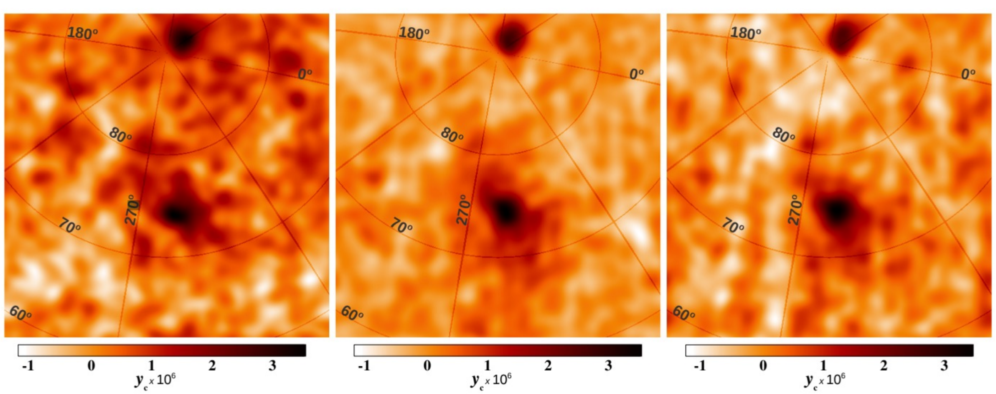
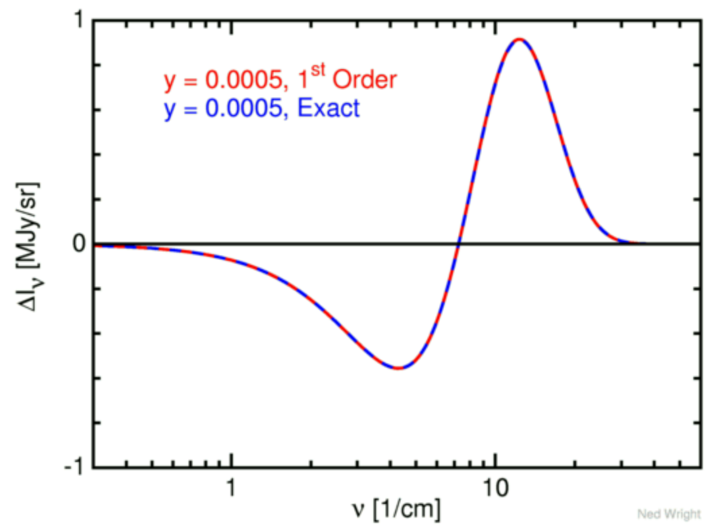


Fig. 4. *Left to right*: individual SZ maps at 70, 100, and 143 GHz in units of 10^6 times the Compton parameter. The field of view is 31°.9 and the coordinates are Galactic. Virgo is the extended source below the centre. The Coma cluster is also clearly visible in the top of the map. The maps have been smoothed to a resolution of 1°.5 to highlight the faintest structures.



Maps of Virgo and Coma clusters from Planck collab. (2015)

http://www.astro.ucla.edu/~wright/SZ-spectrum.html

Propagation of EM radiation in a plasma

Astrophysical gas containing free electrons (e.g. the interstellar medium) is conductive, leading to currents that modify Maxwell's equations

Let's consider the electron equation of motion with a plane-parallel EM wave $\boldsymbol{E} = \boldsymbol{E_0} \exp\big(i(\boldsymbol{k}.\boldsymbol{r} - \omega t)\big)$ and assuming no external *B*-field and $v \ll c$:

$$m_e \dot{\boldsymbol{v}} = -e\boldsymbol{E} \sim -e\boldsymbol{E_0} \exp(i(\boldsymbol{k}.\boldsymbol{r} - \omega t))$$

Writing $\boldsymbol{v} = \boldsymbol{v_0} \exp\big(i(\boldsymbol{k}.\boldsymbol{r} - \omega t)\big)$

we find $\boldsymbol{v_0} = \frac{e}{i\omega m_e}\boldsymbol{E_0}$

Hence the current density is $\boldsymbol{j} = \boldsymbol{j_0} \exp\big(i(\boldsymbol{k}.\boldsymbol{r} - \omega t)\big)$ with

$$\boldsymbol{j_0} = -n_e e \boldsymbol{v_0} = \frac{i n_e e^2}{\omega m_e}\boldsymbol{E_0} = \sigma \boldsymbol{E_0}$$

where σ is the conductivity (pure imaginary and positive multiple of i) ➔ current lags *E*-field)

Propagation of EM radiation in a plasma

Do these electron motions ever lead to a non-zero charge density, ρ

Let's entertain the possibility by writing $\rho = \rho_0 \exp\big(i(\boldsymbol{k}.\boldsymbol{r} - \omega t)\big)$

Then charge conservation tells us

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{j} = 0 \Rightarrow -i\omega\rho_0 + i\boldsymbol{k}.\boldsymbol{j_0} = 0$$

Multiplying by σ, we obtain $i\sigma\boldsymbol{k}.\boldsymbol{j_0} = i\omega\sigma\rho_0 \Rightarrow i\boldsymbol{k}.\boldsymbol{E_0} = i\omega\sigma\rho_0$

Moreover, $\boldsymbol{\nabla}.\boldsymbol{E} = 4\pi\rho \Rightarrow i\boldsymbol{k}.\boldsymbol{E_0} = 4\pi\rho_0$

Combining these two equations in red, we find that
$\rho_0\,(4\pi - i\omega\sigma) = 0 \Rightarrow \rho_0 = 0$

This also implies that the wave is transverse, just as it was in a vacuum

Dispersion relation (i.e relationship between ω and k

Ampere's law is modified by the presence of currents

$\boldsymbol{\nabla}^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -4\pi \boldsymbol{j}/\mathrm{c}$

Substituting the plane-wave solution $\boldsymbol{A} = \boldsymbol{A_0} e^{i(\boldsymbol{k}.\boldsymbol{x}-\omega t)}$ we obtain

$$-k^2 \boldsymbol{A_0} + \frac{\omega^2}{c^2}\boldsymbol{A_0} = -\frac{4\pi}{c}\boldsymbol{j_0} = -\frac{4\pi\sigma}{c}\boldsymbol{E_0} = -\frac{4\pi\sigma}{c}\left(\frac{i\omega}{c}\boldsymbol{A_0} - \boldsymbol{k}\phi_0\right)$$

Hence, equating the components perpendicular to $\boldsymbol{k}$

$$-k^2 + \frac{\omega^2}{c^2} = -\frac{4\pi\sigma}{c}\left(\frac{i\omega}{c}\right) = -\left(\frac{4\pi}{c}\right)\frac{i n_e e^2}{\omega m_e}\left(\frac{i\omega}{c}\right) = \frac{4\pi n_e e^2}{m_e c^2}$$

$$\Rightarrow \omega^2 - k^2 c^2 = \frac{4\pi n_e e^2}{m_e} \equiv \omega_p{}^2$$

where $\omega_p = \sqrt{\frac{4\pi n_e e^2}{m_e}}$ is called the *plasma frequency*

Dispersion relation (i.e relationship between ω and k)

$$\omega_p = \sqrt{\frac{4\pi n_e e^2}{m_e}} = 5.64\times 10^4 \sqrt{\frac{n_e}{\mathrm{cm}^{-3}}}\,\mathrm{rad\ s^{-1}}$$

$$\Rightarrow \qquad \nu_p = \frac{\omega_p}{2\pi} = \sqrt{\frac{n_e e^2}{\pi m_e}} = 9.0\sqrt{\frac{n_e}{\mathrm{cm}^{-3}}}\ \mathrm{kHz}$$

The dispersion relation $\omega^2 - k^2 c^2 = {\omega_p}^2$

$$\Rightarrow k = \frac{\sqrt{\omega^2 - {\omega_p}^2}}{c} \qquad\qquad \omega = \sqrt{{\omega_p}^2 + k^2 c^2}$$

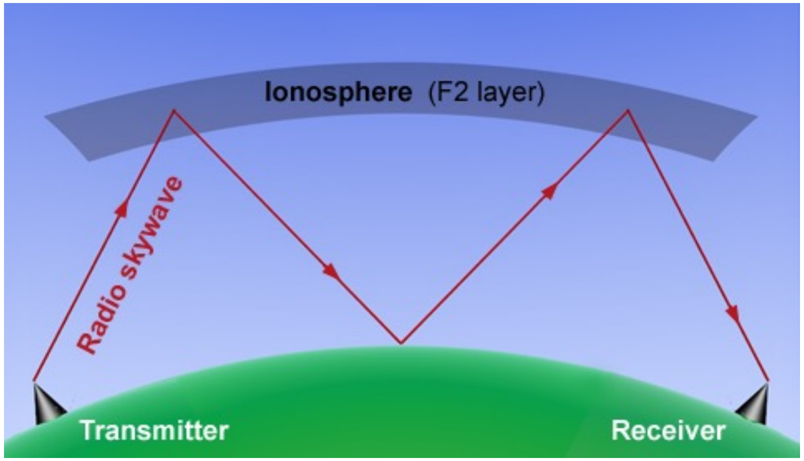
k is only real above the plasma frequency, below which waves cannot propagate

For $\omega < \omega_p$, $k = i\frac{\sqrt{{\omega_p}^2 - \omega^2}}{c}$, an we have an “evanescent” wave in which the amplitude declines exponentially. This leads to reflection at the interface between a region with $\omega > \omega_p$ and $\omega < \omega_p$

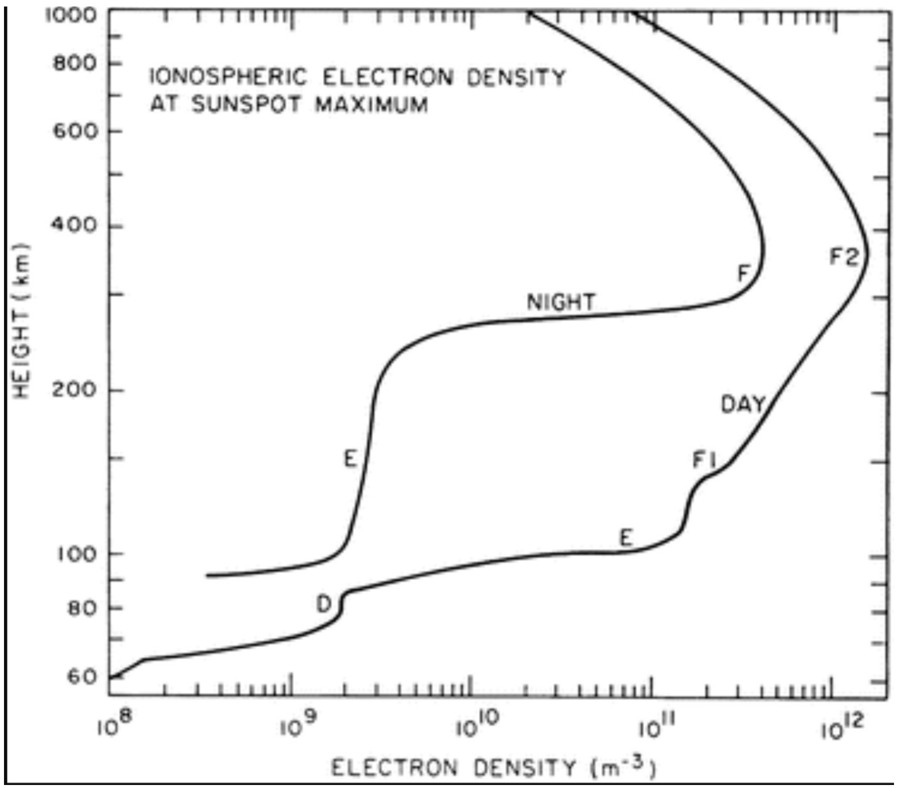
Dispersion relation (i.e relationship between ω and k)

Example: Earth's ionosphere where $n_e \sim\ 10^5 - 10^6\ \mathrm{cm}^{-3}$
$\Rightarrow\ \ \nu_p =\ \text{few MHz}$

AM radio waves (530 – 1600 kHz) are reflected off the ionosphere and can travel large distances bouncing between the Earth and the ionosphere



Sky & Telescope graphic



J. V. Evans and T. Hagfors, *Radar Astronomy*, 1968

FM radio waves (87 – 107 MHz) are not reflected: need line-of-sight to transmitter

Phase and group velocity

When waves travel in a "dispersive" medium (i.e. where ω/k is a function of ω), there are two key velocities

1) Phase velocity

$$v_\phi = \frac{\omega}{k} = \frac{c\omega}{\sqrt{\omega^2 - {\omega_p}^2}} = \left(1 - \frac{{\omega_p}^2}{\omega^2}\right)^{-1/2} c$$

This is the speed ($> c$) at which the peaks of a sine wave move through space

The index of refraction is $n_r \equiv c/\, v_\phi = \left(1 - \frac{\omega^2}{{\omega_p}^2}\right)^{1/2} < 1$

2) Group velocity

$$v_g = \frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}\sqrt{{\omega_p}^2 + k^2c^2} = \frac{kc}{\sqrt{{\omega_p}^2 + k^2c^2}} = \frac{kc}{\omega} = \frac{c^2}{v_\phi} = \left(1 - \frac{{\omega_p}^2}{\omega^2}\right)^{+1/2} c$$

This is the speed ($< c$) at which a wavepacket (pulse) will propagate, and is the speed at which information can travel through a plasma

Pulsar dispersion

The travel time for a pulse emitted by a pulsar at distance *D* is

$$\tau = \int \frac{ds}{v_g} = \int \left(1 - \frac{{\omega_p}^2}{\omega^2}\right)^{-1/2} \frac{ds}{c}$$

$$\sim \int \left(1 + \frac{{\omega_p}^2}{2\omega^2}\right) \frac{ds}{c} = \frac{D}{c} + \int \left(\frac{{\omega_p}^2}{2c\omega^2}\right) ds, \qquad \text{provided } \omega \gg \omega_p$$

There is a frequency-dependent delay,

$$\Delta\tau = \int \left(\frac{{\omega_p}^2}{2c\omega^2}\right) ds = \int \frac{4\pi n_e e^2}{2m_e c\omega^2} ds = \frac{2\pi e^2}{m_e c\omega^2} \int n_e \, ds$$

We can also write this

$$\Delta\tau = \frac{e^2}{2\pi m_e c^3} \lambda^2 N_e = 4.6 \left(\frac{N_e}{\mathrm{cm^{-3}pc}}\right) \left(\frac{\lambda}{\mathrm{cm}}\right)^2 \mu\mathrm{s}$$

where $N_e = \int n_e \, ds$ is the electron column density (sometimes called the dispersion measure)

Pulsar dispersion

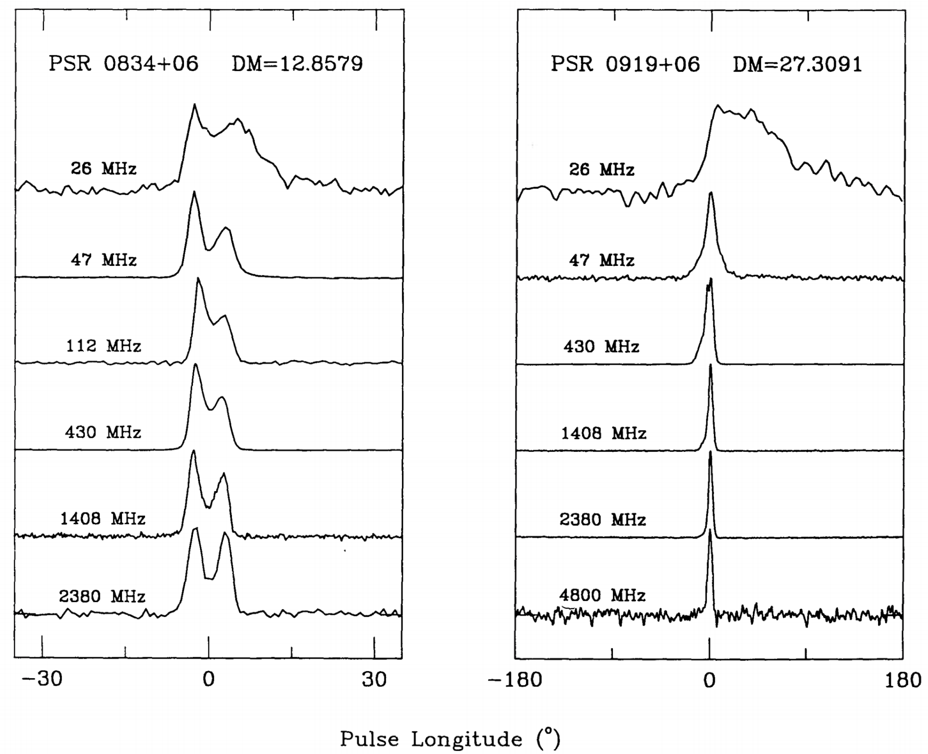
Measurements of pulsar dispersion measure are a key method for determining the electron density in the interstellar medium

If the distances are known (e.g. from trigonometric parallax), we can estimate then mean electron density along the sight line

$$\langle n_e \rangle = \frac{N_e}{D}$$

Typical values are a few x 0.01 cm^{-3} (for sight-lines that do not intersect known HII regions)

“Warm ionized medium” (WIM)



Phillips and Wolszczan 1992

Lecture 21
Plasma effects II / Atoms

Goals: understand

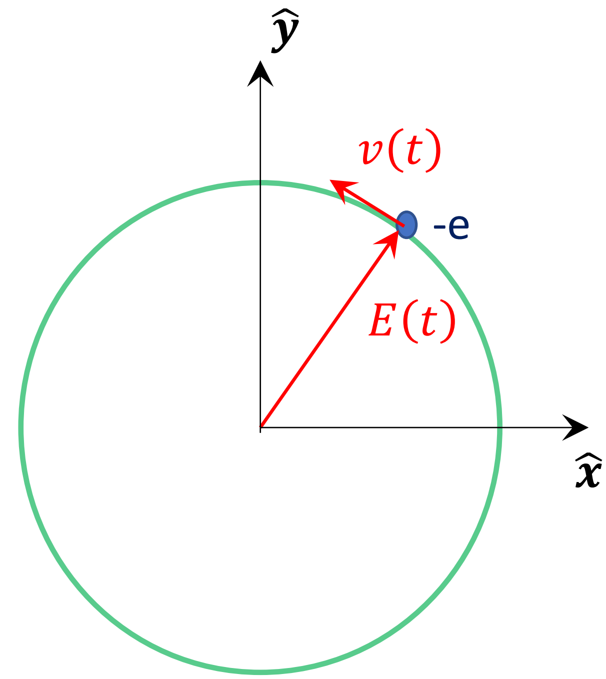
Faraday rotation and its astrophysical applications

Structure of atoms

Effect of interstellar magnetic fields

The dispersion effect we considered first is independent of the polarization state of the wave

But now let's turn to circularly polarized radiation and reconsider the motion of an electron. The electric field associated with the EM wave rotates in a circle:



The electron rotates at speed v in a circle, with a centripetal acceleration of magnitude

$$a = \frac{|v_0|^2}{r} = |v_0|\,\omega = \frac{-eE_0}{m_e}$$

$$E(t) = E_0(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})e^{-i\omega t}$$
$$v(t) = v_0(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})e^{-i\omega t}$$
$$j(t) = j_0(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})e^{-i\omega t}$$

$v(t)$ lags a $\Rightarrow v_0 = \frac{-ieE_0}{m_e\omega} \Rightarrow j_0 = -n_e e v_0 = \frac{in_e e^2 E_0}{m_e\omega} \Rightarrow \sigma = \frac{in_e e^2}{\omega m_e}$

as before

Effect of interstellar magnetic fields

Suppose there is now a fixed interstellar magnetic field $B_\parallel$ along the $\pm$ z-axis. We now have

$$a = |v_0|\omega = -\frac{e}{m_e}\left(E_0 \pm \frac{|v_0|B_\parallel}{c}\right) = -\frac{e}{m_e}E_0 \pm |v_0|\omega_B$$

Hence

$$v_0 = \frac{-ieE_0}{m_e(\omega \pm \omega_B)} \Rightarrow j_0 = \frac{in_e e^2 E_0}{m_e(\omega \pm \omega_B)} \Rightarrow \sigma = \frac{in_e e^2}{m_e(\omega \pm \omega_B)}$$

The dispersion relation becomes

$$-k^2 + \frac{\omega^2}{c^2} = -\frac{4\pi\sigma}{c}\left(\frac{i\omega}{c}\right) = \frac{\omega}{\omega \pm \omega_B}\omega_p{}^2/c^2$$

$$k = \frac{\sqrt{\omega^2 - \omega_p{}^2/(1 \pm \omega_B/\omega)}}{c} \sim \frac{\omega}{c}\left(1 - \frac{\omega_p{}^2}{2\omega^2}\left(1 \pm \frac{\omega_B}{\omega}\right)\right)$$

provided $\omega \gg \omega_p \gg \omega_B$

Effect of interstellar magnetic fields

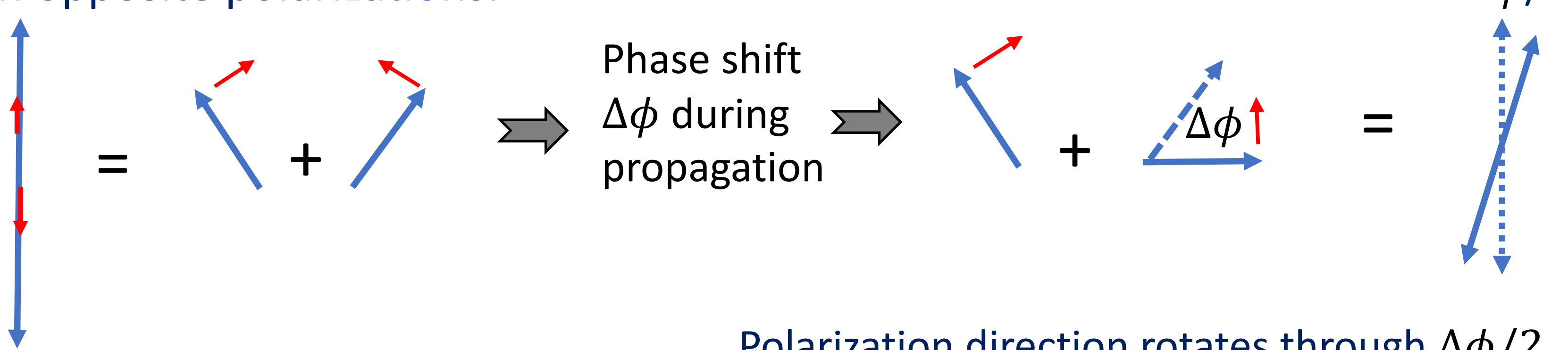
$$k = \frac{\sqrt{\omega^2 - {\omega_p}^2/(1 \pm \omega_B/\omega)}}{c} \sim \frac{\omega}{c}\left(1 - \frac{{\omega_p}^2}{2\omega^2}\left(1 \pm \frac{\omega_B}{\omega}\right)\right)$$

The difference in wavevector for the two opposite circular polarizations is therefore

$$\Delta k = \frac{{\omega_p}^2 \omega_B}{\omega^2 c}$$

The difference in phase is $\Delta\phi = \int \Delta k \, ds$

Consider now the case of a linearly polarized wave (initially with *E* vertical), which can be considered the superposition of two circularly-polarized waves with opposite polarizations.



Polarization direction rotates through $\Delta\phi/2$

Faraday rotation

This phenomenon is called Faraday rotation

The rotation angle is

$$\Delta\theta = \frac{\Delta\phi}{2} = \int \frac{\Delta k}{2} ds = \int \frac{{\omega_p}^2 \omega_B}{2\omega^2 c} ds$$

$$= \int \frac{1}{2\omega^2 c} \frac{4\pi n_e e^2}{m_e} \frac{eB}{m_e c} ds = \frac{\lambda^2 e^3}{2\pi {m_e}^2 c^4} \int n_e B_\parallel ds$$

$$= 8.1\times10^{-4} \left(\frac{\lambda}{\mathrm{cm}}\right)^2 \frac{RM}{\mathrm{cm^{-3} pc\ \mu G}}\ \mathrm{rad}$$

Where $RM = \int n_e B_\parallel ds$ is called the “rotation measure”

Faraday rotation

$$\Delta\theta = \frac{\lambda^2 e^3}{2\pi {m_e}^2 c^4}\int n_e B_\parallel ds = 8.1\times10^{-4}\left(\frac{\lambda}{\mathrm{cm}}\right)^2 \frac{RM}{\mathrm{cm^{-3}\,pc\,\mu G}}\ \mathrm{rad}$$

Notes:

(1) Faraday rotation is only sensitive to the component of $\boldsymbol{B}$ along the line of sight. The component in the "plane of the sky" is irrelevant since for that component the time-averaged $\boldsymbol{v}\times\boldsymbol{B}$ is zero

(2) The sign of the rotation angle depends on the *sense* of $B_\parallel$ (whether towards or away from us). A changing sense along the sight-line can lead to cancellation that reduces the rotation measure.

(3) The ratio of the rotation measure $\int n_e B_\parallel ds$ to the dispersion measure $\int n_e ds$ yields an estimate of the typical magnetic field (although note point (2) above)

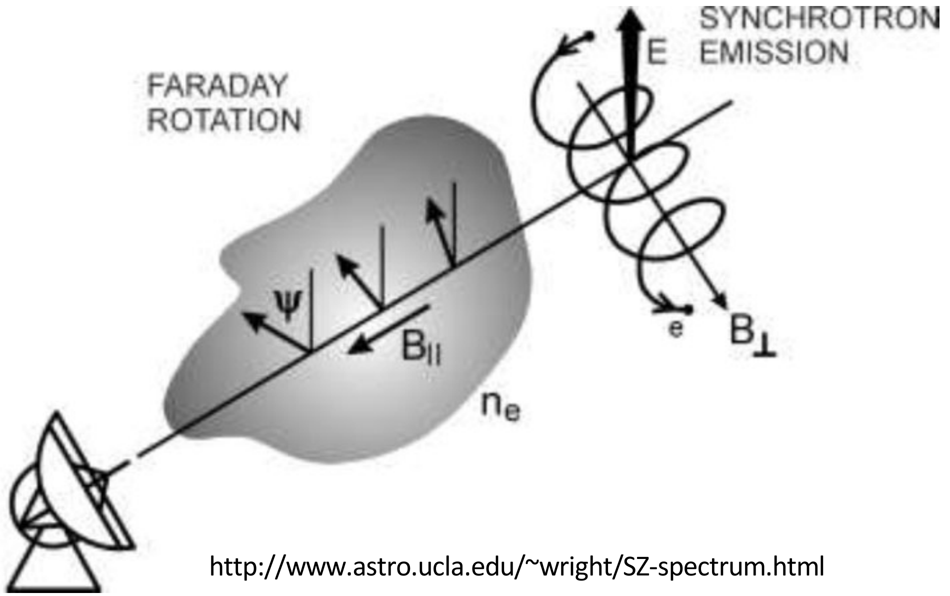
Faraday rotation

$$\Delta\theta = \frac{\lambda^2 e^3}{2\pi {m_e}^2 c^4} \int n_e B_{\parallel} ds = 8.1\times10^{-4} \left(\frac{\lambda}{\mathrm{cm}}\right)^2 \frac{RM}{\mathrm{cm^{-3}pc\ \mu G}}\ \mathrm{rad}$$

Notes:

(4) If there is a significant RM from the front of a synchrotron emission to the back, Faraday rotation mixes radiation with different emergent polarization directions => decrease in overall polarization fraction

This effect is known as *Faraday depolarization*



http://www.astro.ucla.edu/~wright/SZ-spectrum.html

Interaction of radiation with atoms: astrophysical motivation

A) Bound-bound transitions of atoms (and atomic ions) play a critical role in astrophysics as

1) coolants of gas (except at very high temperature)
2) a source of opacity in stars
3) diagnostics of abundances, redshifts, density, temperature

B) Bound-free/free-bound transitions determine the ionization state of astrophysical gas through

1) Photoionization: $\mathrm{X}^{+n} + h\nu \rightarrow \mathrm{X}^{+(n+1)} + e$

Rate per unit volume $= \int \frac{4\pi J_\nu}{h\nu} {\sigma_\nu}^{pi}\, n(\mathrm{X}^{+n}) d\nu$

2) Radiative recombination: $\mathrm{X}^{+(n+1)} + e \rightarrow \mathrm{X}^{+n} + h\nu$

Rate per unit volume $= \alpha_R(T)\, n_e n\left(\mathrm{X}^{+(n+1)}\right)$

Energy levels for atoms: hydrogen (and H-like)

Simplest case: only one electron

The state of the electron is described by five quantum numbers:

n = principal quantum number, which ranges from 1 to ∞

l = orbital angular momentum (in units of $\hbar$), which ranges from 0 to $n - 1$ and is coded with the letters s, p, d, f, g, h…

m_l = azimuthal quantum number, which ranges from $-l$ to l and is the projection of the orbital angular momentum onto some axis

s = ½ is the electronic spin

m_s= $\pm$ ½, is the projection of the spin onto some axis

Energy levels for atoms: hydrogen (and H-like)

The eigenstates are solutions to the time independent Schrodinger equation

$$H\psi = E\psi$$

For a simple Coulomb potential, the Hamiltonian H is

$$H = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{Ze^2}{r}$$

And the energy of system depends only on n (neglecting the effects of spin and quantum electrodynamics)

$$E_n = -Z^2\frac{Ry}{n^2}$$

Energy levels for atoms: hydrogen (and H-like)

$$E_n = -Z^2 \frac{Ry}{n^2}$$

Here the $Ry = e^2/2a_0$ ~ 13.59 eV is called the "Rydberg"

$$a_0 = \frac{\hbar^2}{m_e e^2} = \left(\frac{e^2}{\hbar c}\right)^{-2} \frac{e^2}{m_e c^2} = \alpha^{-2} r_0 = 0.529\ \times 10^{-8}\mathrm{cm}$$ is the Bohr radius,

and $\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$ is the fine-structure constant

Note: if we measure length in units of a_0, and energy in units of e^2/a_0 (sometimes called atomic units), the Hamiltonian becomes dimensionless and can be written

$$H = -\frac{1}{2}\nabla^2 - \frac{Z}{r}$$

Hence $Ry = \frac{m_e e^4}{2\hbar^2} = \frac{1}{2}\alpha^2\, m_e c^2$

Actually, this expression assumes that the nuclear mass $m_N \gg m_e$ so the center of the nuclear potential is coincident with the center of mass. To be more precise, we need to replace m_e with the "reduced mass" $\mu = m_e\, m_N\, /\, (m_e + m_N)$

Lecture 22
Atoms

Goals: understand

Multielectron atoms

Spin-orbit coupling

Multielectron atoms

The energy levels of multielectron atoms are far more complex. The crudest description of a state involves simply specifying the number of electrons in each "orbital" (defined by specific values of n and l.)

Example: the ground state of atomic carbon is $1s^2\ 2s^2\ 2p^2$

This is called the electronic *configuration*

Pauli exclusion principle: maximum number of electrons in state nl is number of m_s values × number of m_l values
= $(2s+1)(2l+1) = 2(2l+1)$

i.e. $2, 6, 10, 14$, ... for ns, np, nd, nf

Multielectron atoms

C $1s^2\,2s^2\,2p^2$

In this case, the $1s^2$ electrons have zero orbital angular momentum are paired with opposite spins so their total spin and orbital angular momentum are both zero. The same is true of the $2s^2$ electrons.

But the $2p^2$ electrons each have $l = 1$ and $s = \frac{1}{2}$. The total orbital angular momentum L therefore ranges from zero to 2 (i.e. can be 0, 1 or 2), and the total spin S can be zero or 1.

A set of states with given values of L and S is called a *term*

Terms are represented with the notation ${}^{2S+1}L$ where L is a *capital* letter in the sequence S, P, D, F, G… for $L = 0, 1, 2, 3, 4$ ….. .

Multielectron atoms

For carbon in the ground state configuration ($1s^2\ 2s^2\ 2p^2$) there are three terms: ${}^3P$, ${}^1D$, and ${}^1S$.

In this particular case (both p electrons in n = 2), not all possible combinations of L and S are permitted by the Pauli exclusion principle. (Thus there are no ${}^1P$, ${}^3D$, or ${}^3S$ terms for this configuration.)

The superscripted symbols $2S + 1$ are the spin-degeneracies, so it is conventional when speaking to refer to ${}^3P$ as "triplet-P" not "three-P" and ${}^1S$ as "singlet-S" not "one-S"

Hamiltonian for multielectron atoms

Just including electrostatic energies (i.e. neglecting the effects of spin), the Hamiltonian H is

$$H = \sum_j \left(-\frac{\hbar^2}{2m_e} \nabla_j^{\,2} - \frac{Ze^2}{r_j} \right) + \sum_{i>j} \frac{e^2}{r_{ij}}$$

where the indices i and j number the electrons

Here r_j is the distance of the jth electron from the nucleus and r_{ij} is the separation between the jth and ith electrons

Because of the $\sum_{i>j} \frac{e^2}{r_{ij}}$ term, the different terms have significantly different energies by $\sim$ 1 eV and transitions between them typically yield visible/near-IR photons

Hamiltonian for multielectron atoms

Terms with larger spin S have a greater degree of spin alignment. The Pauli exclusion principle therefore tends to make the electrons stay further apart, which reduces $\sum_{i>j} \frac{e^2}{r_{ij}}$

➔ larger S states have lower energy (Hund's rule #1)

For this reason, the ${}^3P$ term is the ground state of atomic carbon

The same is true of terms with larger L ➔ larger L states have lower energy (Hund's rule #2)

Thus the ${}^1D$ state is next in energy, and then the ${}^1S$ state

Spin-orbit coupling

Within a given term, there are various possible orientations of L and S. These have slightly different energies, because the spin is associated with a magnetic dipole moment and the electron sees a magnetic field as it moves through the Coulomb potential.

This adds a small additional term to the Hamiltonian

$$H = \sum_j \left(-\frac{\hbar^2}{2m_e} \nabla_j{}^2 - \frac{Ze^2}{r_j} \right) + \sum_{i>j} \frac{e^2}{r_{ij}} + H_{so}$$

where $H_{so} = \xi \boldsymbol{L}.\boldsymbol{S}$

Spin-orbit coupling

If we define the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$, we find that $\boldsymbol{J}^2 = \boldsymbol{L}^2 + \boldsymbol{S}^2 + \boldsymbol{2L.S}$

➔ $H_{so} = \xi \boldsymbol{L.S} = \frac{\xi}{2}(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2)$

$$\langle H_{so} \rangle = C[J(J+1) - L(L+1) - S(S+1)]$$

For a given term, L and S are fixed but the energy depends on J. Thus the term may be split into several of different J, which are indicated using the notation ${}^{2S+1}L_J$, where J ranges from $|L - S|$ to $L + S$

This splitting is called fine structure

Spin-orbit coupling

Example: ground state term of atomic carbon, ${}^3P$, splits into ${}^3P_0$, ${}^3P_1$ and ${}^3P_2$

$$\langle H_{so} \rangle = C[J(J+1) - L(L+1) - S(S+1)]$$

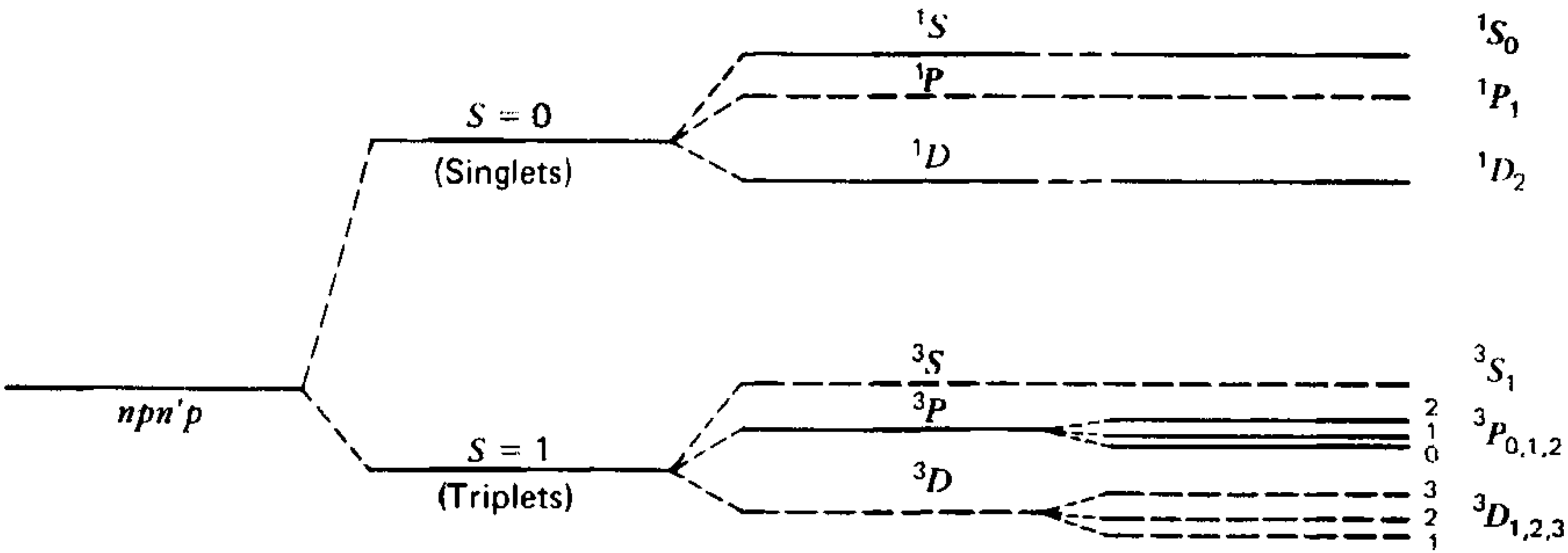
C is positive when a shell (e.g. $2p$) is less than half-full (e.g. Carbon $1s^2\,2s^2\,2p^2$), but negative when a shell is more than half-full (e.g. Oxygen $1s^2\,2s^2\,2p^4$)

For negative C, the energy is a decreasing function of J and the term is called *inverted* (as opposed to "normal")

(Hund's rule #3)

Hierarchy of energy splittings

Degeneracy=2J+1



$^1S_0$	1
$^1P_1$	3
$^1D_2$	5
$^3S_1$	3
$^3P_{0,1,2}$	1,3,5
$^3D_{1,2,3}$	3,5,7

Each electron has 2 possible values of m_s and 3 possible values of m_l ➔ 6 possible combinations

Sum = 36 = 6 x 6 if n and n' differ

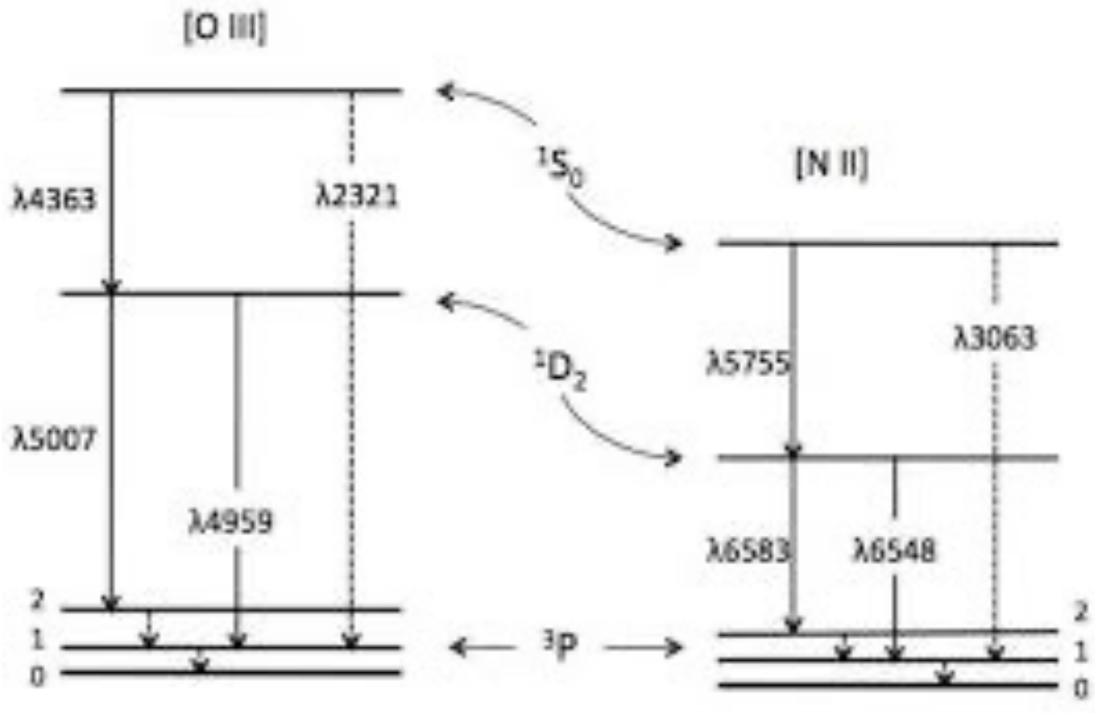
(*b*)

Figure 9.2b ***Same as a, but for two p electrons. Dashed levels are absent from the multiplet if the electrons are equivalent ($n = n'$). (Taken from Leighton, R., 1959, Principles of Modern Physics, McGraw-Hill, New York.)***

Astrophysical importance of fine structure

1) Optical line emission between terms can be split into multiple nearby lines

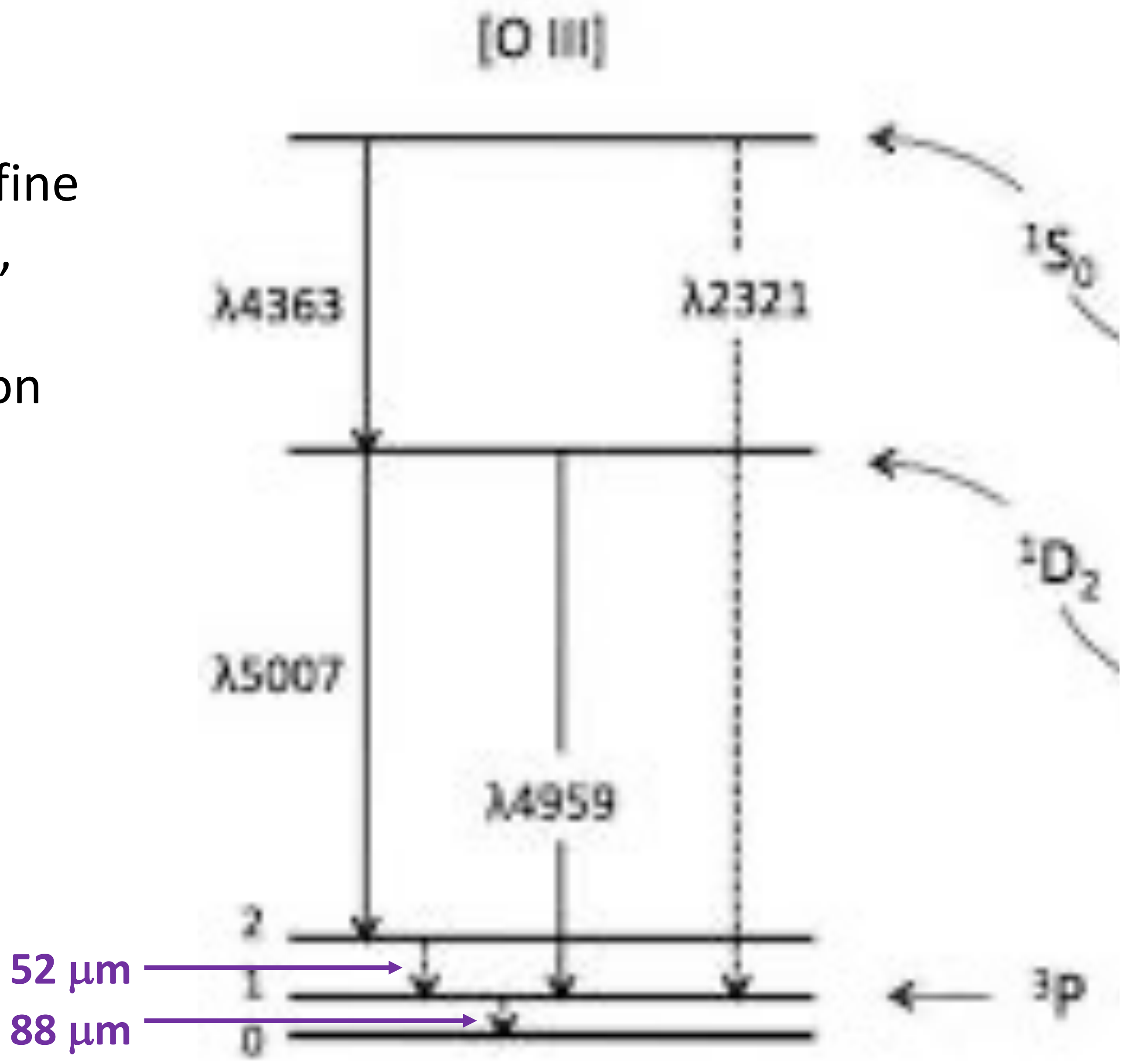
   Example: [OIII] and [NII] lines observed from HII regions (both $1s^2\, 2s^2\, 2p^2$)



Forbidden lines of O^{++} Forbidden lines of N^{+}

Astrophysical importance of fine structure

2) Transitions can occur *between* fine structure states, leading to far-infrared radiation



Lecture 23
Atoms II

Goals: understand

Parity

Hyperfine structure

Radiative transitions and selection rules

Parity

One final note about the electronic state of an atom

Every state has an overall wavefunction that is either symmetric or antisymmetric under mirror reflection

i.e. if we make the transformation $x \rightarrow -x, \quad y \rightarrow -y, \quad z \rightarrow -z$
then either $\psi \rightarrow -\psi$ (odd parity) or $\psi \rightarrow +\psi$ (even parity)

The parity depends solely on whether $\sum l$ is even ($\rightarrow$ even parity) or odd ($\rightarrow$ odd parity)

Thus all terms in a given configuration have the same parity. An odd parity is indicated by a superscripted O after the term
e.g. the ground state term of N ($1s^2\ 2s^2\ 2p^3$) is ${}^4S^O$

Hyperfine splitting

When the *nucleus* has non-zero spin, $I \neq 0$ there is an additional splitting that occurs for electronic states with $J \neq 0$

Abundant nuclei with non-zero spin (in parentheses):
^{1}H (½), ^{2}H (1), ^{14}N (1), ^{23}Na (3/2), ^{35}Cl(3/2), ^{37}Cl(3/2), ^{13}C (½)

(dominant isotopes in black, others in purple)

NB: α-nuclei with even number of neutrons = number of protons are typically spinless (e.g. ^{4}He, ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg, ^{28}Si, ^{32}S, ^{36}Ar, ^{40}Ca)

The vector sum of the nuclei spin and electronic angular momentum is given the quantum number $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{J}$

The interaction between the nuclear magnetic dipole and the electronic magnetic dipole leads to a (very small) splitting

Fine structure of C^+ ground state term

Example 1: ground state term of C^+. The configuration is $1s^2\,2s^2\,2p$

Question 1: what terms are possible with this configuration?

Fine structure of C^+ ground state term

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Question 1: what terms are possible with this configuration?

Answer: there only one term, with $S = \frac{1}{2}, L = 1$, i.e. ${}^2P$

Fine structure of C$^+$ ground state term

Example 1: ground state term of C$^+$. The configuration is $1s^2\ 2s^2\ 2p$

Question 1: what terms are possible with this configuration?

Answer: there only one term, with $S = \frac{1}{2}, L = 1$, i.e. ${}^2P$

Question 2: what does spin-orbit coupling do to this term?

Fine structure of C$^+$ ground state term

Example 1: ground state term of C$^+$. The configuration is $1s^2\,2s^2\,2p$

Question 1: what terms are possible with this configuration?

Answer: there only one term, with $S = \frac{1}{2}, L = 1$, i.e. ^2P

Question 2: what does spin-orbit coupling do to this term?

Answer: it splits it into two fine-structure states $^2P_{1/2}$ and $^2P_{3/2}$

$$\begin{aligned} \Delta E &= 7.86\text{meV} \\ &= 1900.536\text{GHz} \\ &= 157.7\mu\text{m} \end{aligned}$$

$^2P_{3/2}$

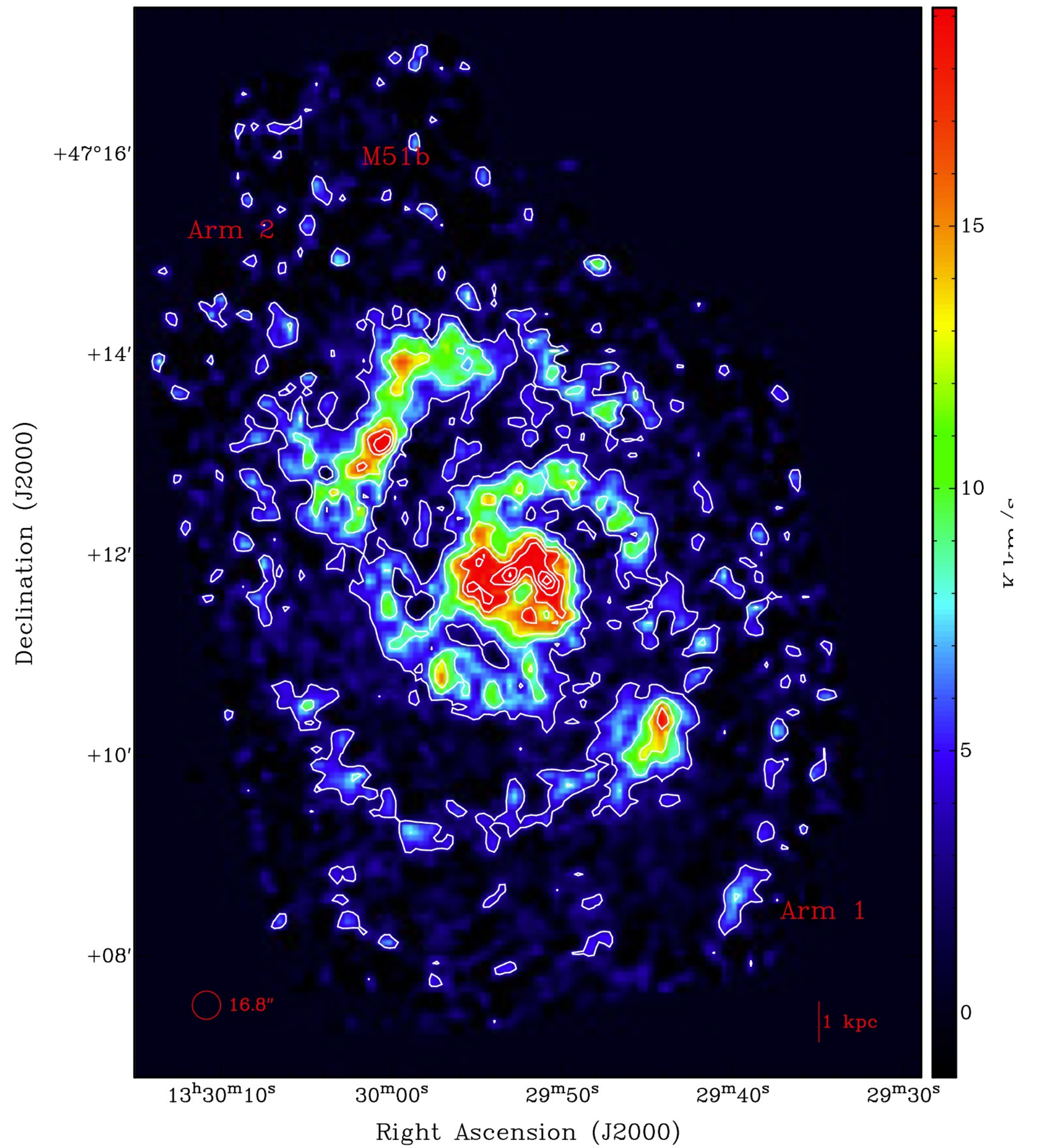
$^2P_{1/2}$

These have an energy difference leading to a fine-structure transition near 158 μm. This is typically the most luminous spectral line emitted by galaxies, because it dominates the cooling of cold interstellar gas

Fine structure of C⁺ ground state term

[CII] map of the face-on spiral galaxy M51, from Pineda et al. 2020

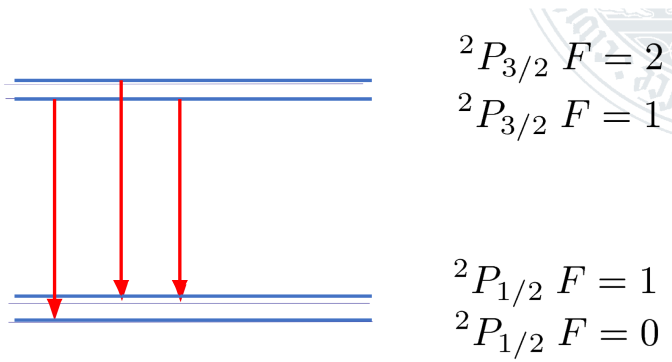
Obtained with the upGREAT instrument on the SOFIA airborne observatory

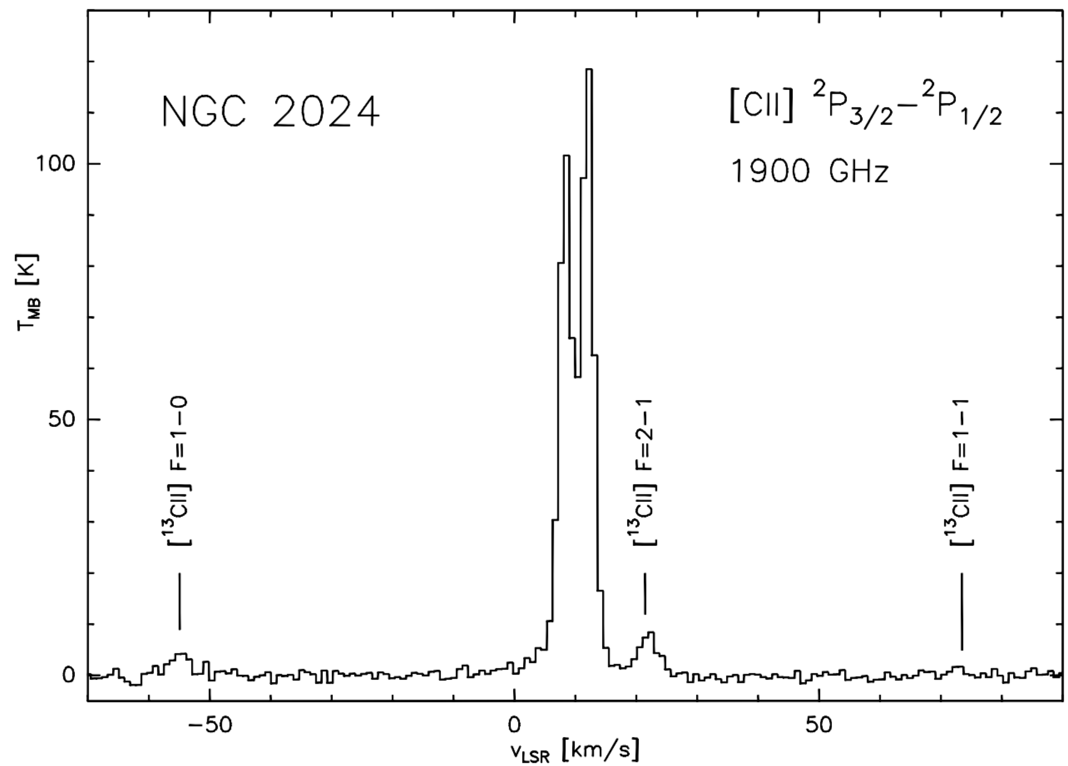


Hyperfine splitting

But suppose we now have $^{13}C^+$, with nuclear spin ½ (relative isotopic abundance ~ 1%)

Each fine-structure state is split into two and there are three slightly-separated transitions (F = 2 – 0 is forbidden)

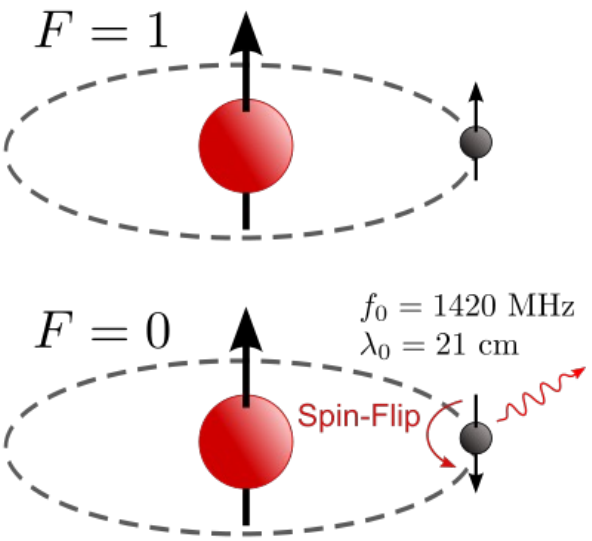


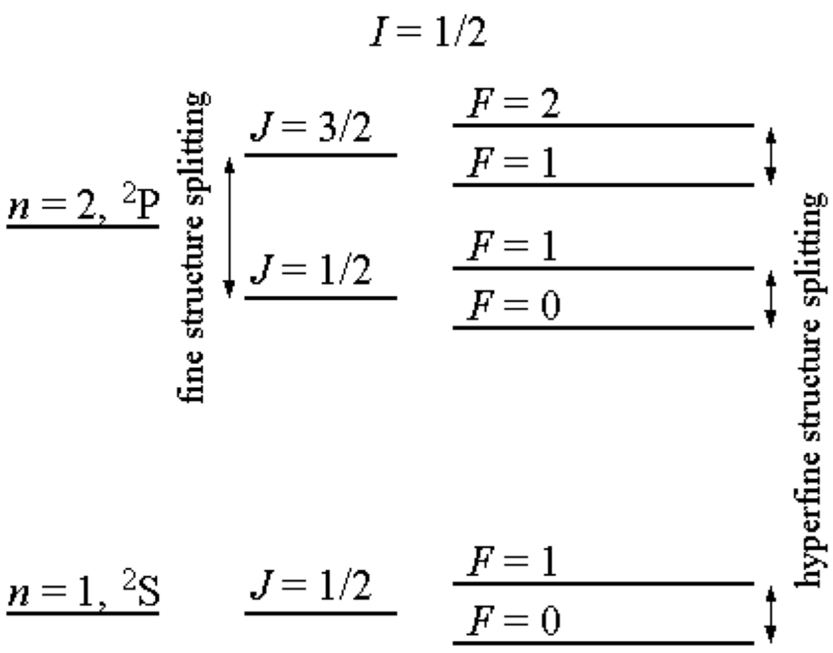


Hyperfine splitting

Example 2: ground state term of atomic hydrogen, ${}^2S_{1/2}$, is split into two states: F = 1 and F = 0. Transitions *between* these two states result in a photon at 1,420,405,751.7667 ± 0.0009 Hz, which is equivalent to 21.1061140542 cm

The key spectral line for studying cold neutral gas in the Universe





Radiative transitions

The rate of radiative transitions between any two states is related to the wavefunctions of the initial and final states, ψ_i and ψ_j

In the dipole approximation, the Einstein-A coefficient for two non-degenerate states is given by

$$A_{ij} = \frac{64\pi^4 \upsilon^3}{3hc^3} \langle \psi_i | (-e\boldsymbol{r}) | \psi_f \rangle^2$$

Here $\langle \psi_i | (-e\boldsymbol{r}) | \psi_f \rangle \equiv \int \psi_i (-e\boldsymbol{r}) \psi_j{}^* \, d^3\boldsymbol{r}$ is the transition dipole moment $\boldsymbol{d_{ij}}$

The dipole approximation, which is based on the approximation $e^{i\boldsymbol{k}.\boldsymbol{r}} \sim 1$, (i.e. $\lambda \gg a_0$) is generally very good provided $\boldsymbol{d_{ij}} \neq 0$. Transitions with $\boldsymbol{d_{ij}} \neq 0$ are called "dipole-allowed."

In the case of "forbidden" transitions with $\boldsymbol{d_{ij}} = 0$, additional terms in the expansion of $e^{i\boldsymbol{k}.\boldsymbol{r}}$ must be included. A_{ij} is not necessarily zero, but is typically much smaller than in a dipole-allowed transition.

Selection rules for dipole-allowed transitions

"Laporte's rule":

$\boldsymbol{d_{ij}} = \int \psi_i\,(-e\boldsymbol{r})\psi_j{}^*\, d^3\boldsymbol{r}$ can only be non-zero if the integrand has even parity.

$\boldsymbol{r}$ has odd parity, so the product $\psi_i\,\psi_j{}^*$ needs to be odd

In other words, for non-zero $\boldsymbol{d_{ij}}$, one of the wavefunctions must be odd and the other must be even ➔ the parity has to change

The parity is the same for all states within a given configuration (since $\sum l$ is the same)

➔ All transitions between different terms within a given configuration are forbidden by Laporte's rule, as are all fine-structure transitions

This is indicated with the use of square brackets around the designation of the ion (e.g. [OIII] means a forbidden transition of the O^{++} ion)

Selection rules for dipole-allowed transitions

Selection rules related to angular momentum

For dipole-allowed transitions, the additional selection rules are
$\Delta J = 0$ or ± 1 (except that $J = 0 \rightarrow 0$ is forbidden)
$\Delta L = 0$ or ± 1
$\Delta S = 0$ (or else the transition is called "spin-forbidden")

The first rule can be understood in terms of angular momentum conservation

The angular momentum of the emitted photon is $\hbar$, since the photon is a spin 1 particle. There could also be a component due to the photon's linear motion, but this is of order $a_0 p = a_0 h\nu/c$ = $(2\pi a_0/\lambda)\hbar \ll \hbar$

Thus, the final angular momentum is the vector sum of $\hbar$ and $J_f\hbar$, and this must equal the initial angular momentum of the atom, $J_i\hbar$

For $J_f \neq 0$, this final angular momentum ranges from $(J_f - 1)\hbar$ to $(J_f + 1)\hbar$, requiring $\Delta J = 0$ or ± 1

For $J_f = 0$, the final angular momentum is just $\hbar$, requiring $J_i = 1$ and disallowing $J = 0 \rightarrow 0$

NIST database of atomic transitions (wavelengths and A-values)

https://www.nist.gov/pml/atomic-spectra-database

Atomic Spectra Database

NIST Standard Reference Database 78

Version 5.8

Last Update to Data Content: October 2020 | Version History & Citation Information | Disclaimer |
DOI: https://dx.doi.org/10.18434/T4W30F

Welcome to the NIST Atomic Spectra Database, NIST Standard Reference Database #78. The spectroscopic data may be selected and displayed according to wavelengths or energy levels by choosing one of the following options:

LINES	Spectral lines and associated energy levels displayed in wavelength order with all selected spectra intermixed or in multiplet order. Transition probabilities for the lines are also displayed where available.
LEVELS	Energy levels of a particular atom or ion displayed in order of energy above the ground state.
GROUND STATES & IONIZATION ENERGIES	Ground states and ionization energies of atoms and atomic ions.
LIBS	ASD Interface for Laser Induced Breakdown Spectroscopy (LIBS)

NIST database of atomic transitions (wavelengths and A-values)

Allows searches by ion, wavelength range, upper and lower state energies

Lists quantum numbers and energies for the upper and lower states, wavelengths, and Einstein A-coefficients

Example search

The type of transition is listed with the following (conventional) coding

E1: electric dipole (i.e. dipole-allowed according to the selection rules discussed previously)

Typical A_{ij} of 10^9 s^{-1} for visible wavelength transitions

M1: magnetic dipole (no parity change; otherwise the same selection rules as E1)
Typically five orders of magnitude lower A_{ij} than E1

E2: electric quadrupole (no parity change, $|\Delta J|$ up to 2). Even lower $A_{ij} \propto \upsilon^5$
(Of little/no astrophysical importance)

Lecture 24
Molecules

Goals: understand

Importance of molecules in astrophysics

Born-Oppenheimer approximation

Electronic, vibrational and rotational transitions

Motivation for studying molecules

1) Molecules are ubiquitous

A wide variety of molecules are found in a wide variety of astrophysical environments:

- Interstellar medium – sites of star formation
- Circumstellar outflows
- Cometary comae
- Accretion disks
- High-z galaxies
- Stellar and planetary atmospheres

List of ~ 200 molecules detected in the ISM: some familiar, some very exotic

Cologne Database for Molecular Spectroscopy (CDMS) list of interstellar and circumstellar molecules

2 atoms

H_2, AlF, AlCl, C_2**, CH, CH^+, CN, CO, CO^+, CP, SiC, HCl, KCl, NH, NO, NS, NaCl, OH, PN, SO, SO^+, SiN, SiO, SiS, CS, HF, HD, FeO ?, O_2, CF^+, SiH ?, PO, AlO, OH^+, CN^-, SH^+, SH, HCl^+, TiO, ArH^+, NO^+ (?)

3 atoms

C_3 *, C_2H, C_2O, C_2S, CH_2, HCN, HCO, HCO^+, HCS^+, HOC^+, H_2O, H_2S, HNC, HNO, MgCN, MgNC, N_2H^+, N_2O, NaCN, OCS, SO_2, c-SiC_2, CO_2 *, NH_2, H_3^+ (*), SiCN, AlNC, SiNC, HCP, CCP, AlOH, H_2O^+, H_2Cl^+, KCN, FeCN, HO_2, TiO_2, C_2N, Si_2C

4 atoms

c-C_3H, l-C_3H, C_3N, C_3O, C_3S, C_2H_2 *, NH_3, HCCN, $HCNH^+$, HNCO, HNCS, $HOCO^+$, H_2CO, H_2CN, H_2CS, H_3O^+, c-SiC_3, CH_3 *, C_3N^-, PH_3, HCNO, HOCN, HSCN, H_2O_2, C_3H^+, HMgNC, HCCO

5 atoms

C_5 *, C_4H, C_4Si, l-C_3H_2, c-C_3H_2, H_2CCN, CH_4 *, HC_3N, HC_2NC, HCOOH, H_2CNH, H_2C_2O, H_2NCN, HNC_3, SiH_4 *, H_2COH^+, C_4H^-, HC(O)CN, HNCNH, CH_3O, NH_4^+, H_2NCO^+ (?), $NCCNH^+$

6 atoms

C_5H, l-H_2C4, C_2H_4*, CH_3CN, CH_3NC, CH_3OH, CH_3SH, HC_3NH^+, HC_2CHO, NH_2CHO, C_5N, l-HC_4H *, l-HC_4N, c-H_2C_3O, H_2CCNH, C_5N^-, HNCHCN

7 atoms

C_6H, CH_2CHCN, CH_3C_2H, HC_5N, CH_3CHO, CH_3NH_2, c-C_2H_4O, H_2CCHOH, C_6H^-, CH_3NCO

8 atoms

CH_3C_3N, $HC(O)OCH_3$, CH_3COOH, C_7H, C_6H_2, CH_2OHCHO, l-HC_6H *, CH_2CHCHO (?), CH_2CCHCN, H_2NCH_2CN, CH_3CHNH

9 atoms

CH_3C_4H, CH_3CH_2CN, $(CH_3)_2O$, CH_3CH_2OH, HC_7N, C_8H, $CH_3C(O)NH_2$, C_8H^-, C_3H_6, CH_3CH_2SH (?)

10 atoms

CH_3C_5N, $(CH_3)_2CO$, $(CH_2OH)_2$, CH_3CH_2CHO, CH_3CHCH_2O

11 atoms

HC_9N, CH_3C_6H, C_2H_5OCHO, $CH_3OC(O)CH_3$

12 atoms

c-C_6H_6 *, n-C_3H_7CN, i-C_3H_7CN, $C_2H_5OCH_3$ (?)

> 12 atoms

$HC_{11}N$, C_{60} *, C_{70} *, C_{60}^+ *

*vibrational spectra only **electronic spectra only (updated Aug. 2016)

Motivation for studying molecules

2) Molecules as probes

a) As a probe of excitation conditions: thanks to the rich spectrum of rotational, vibrational and electronic transitions

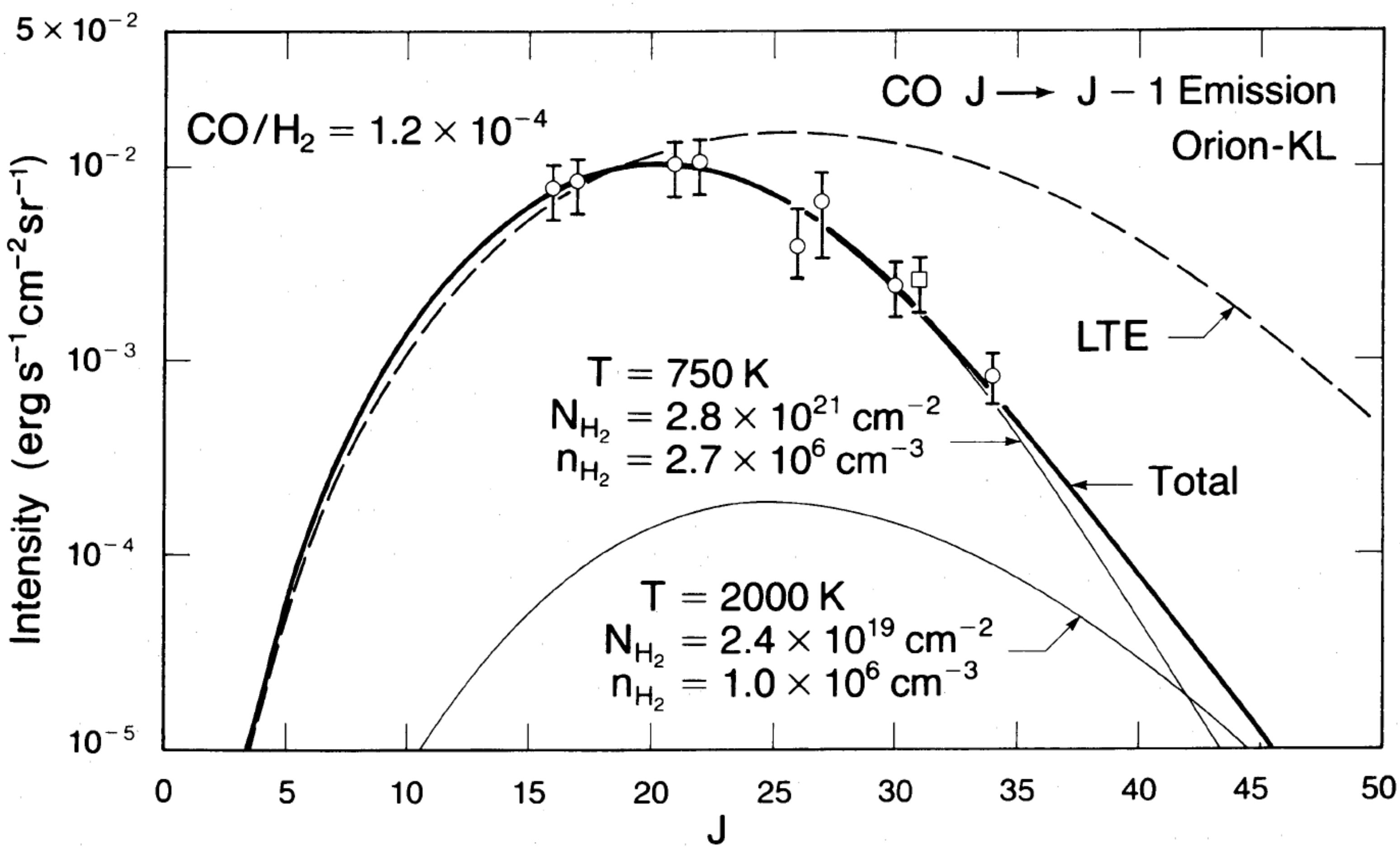
b) As a kinematic probe: thanks to the high brightness temperature of maser spots

c) As a chemical probe

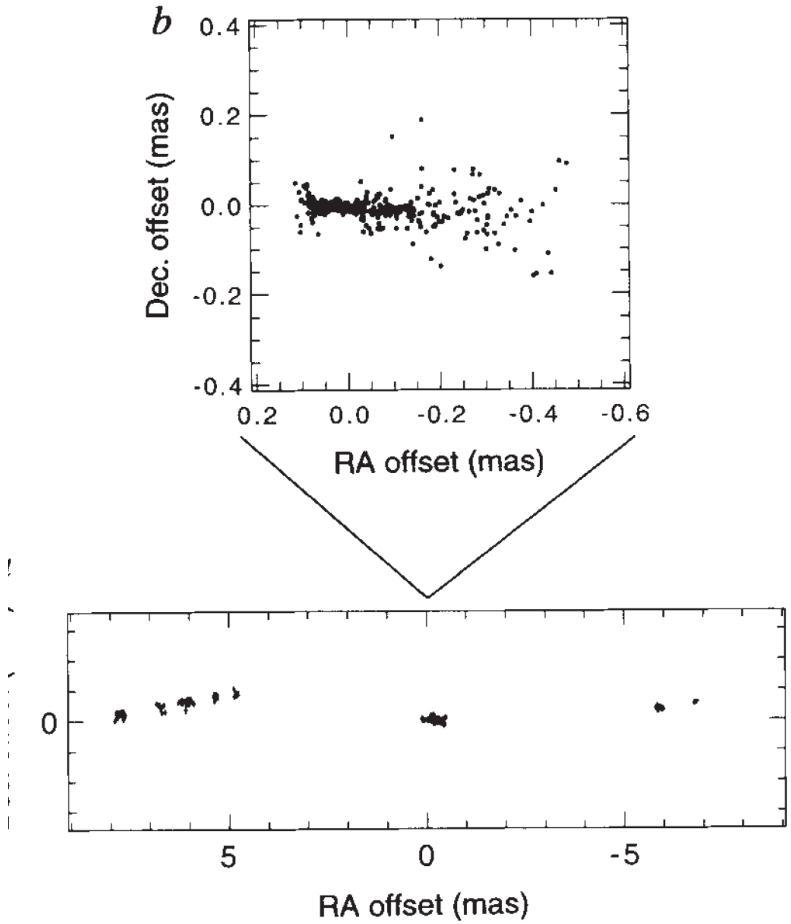
d) As a probe of isotopic abundances: thanks to the large isotopic shift

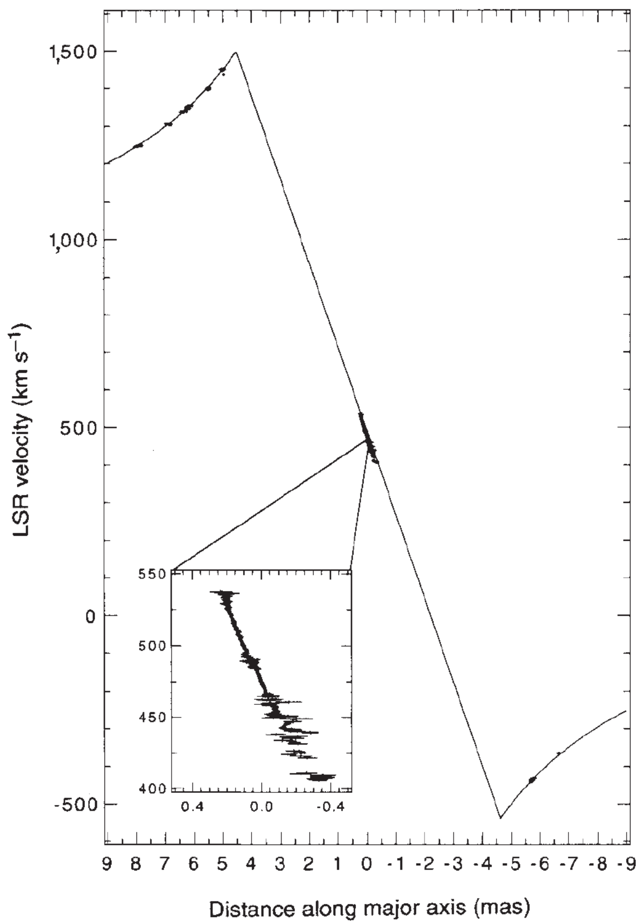
e) As a magnetic probe: thanks to the Zeeman shift

Excitation conditions probed by observations of CO rotational lines (Watson et al. 1985, ApJ)



Masers as a kinematic probe of circumnuclear gas in AGN
(Miyoshi et al. 1995, Nature)





Molecules as a probe of isotopic abundances

(from Kahane et al. 1992, ApJ)

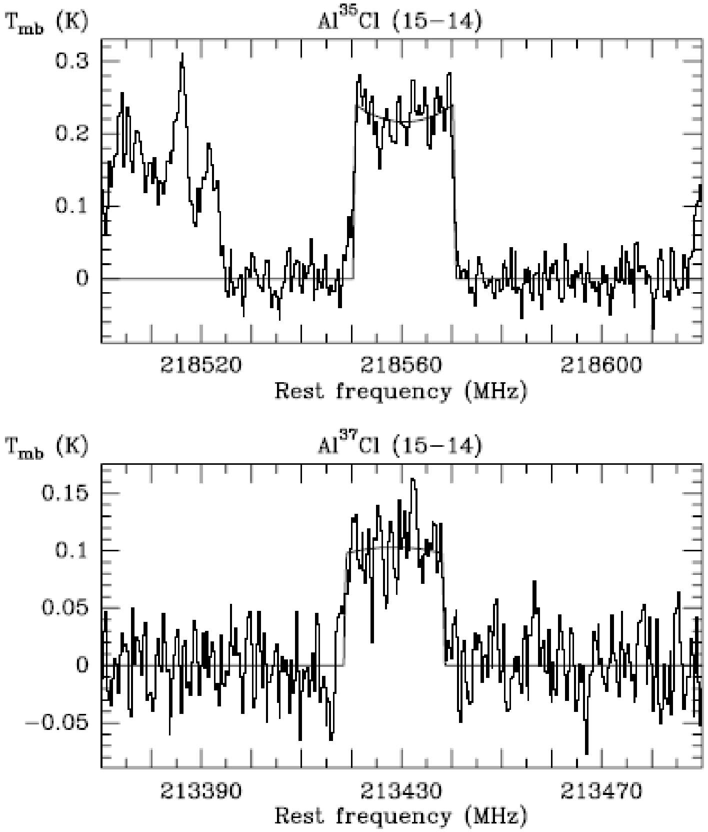


Fig. 2. The same as Fig. 1 for the AlCl (15-14) lines. The spectral resolutions are respectively 0.43 and 0.44 km s^{-1}.

Table 2. Observed isotopic ratios towards IRC+10216.

Ratio	Value	1 σ	Ref.[a]	Solar[b]
$Na^{35}Cl/Na^{37}Cl$ (7-6)	2.33	0.50	(1)	
$Al^{35}Cl/Al^{37}Cl$ (15-14)	2.15	0.33	(1)	
$Na^{35}Cl/Na^{37}Cl$ (8-7)	1.78	0.59	(2)	
$Al^{35}Cl/Al^{37}Cl$ (10-9)	3.17	0.79	(3)	
$Al^{35}Cl/Al^{37}Cl$ (11-10)	2.40	0.76	(3)	
$^{35}Cl/^{37}Cl$[c]	2.30	0.24	(1)	3.13
$^{12}C/^{13}C$	45	3	(3)	89
$^{14}N/^{15}N$	$>$ 4400		(4)	270
$^{16}O/^{17}O$	840	200	(5)	2610
$^{16}O/^{18}O$	1260	280	(5)	499
$^{24}Mg/^{25}Mg$	7.60	1.1	(6)	7.94
$^{24}Mg/^{26}Mg$	6.50	0.7	(6)	7.19
$^{29}Si/^{30}Si$	1.45	0.13	(3)	1.52
$^{28}Si/^{29}Si$[d]	$>$ 15.4		(3)	19.8
$^{34}S/^{33}S$	5.55	0.31	(3)	5.62
$^{32}S/^{34}S$	21.8	2.6	(3)	22.5

[a] the references are the following: (1) this paper; (2) Cernicharo et al. 1987; (3) Cernicharo et al. 2000; (4) Kahane et al. 1988; (5) Kahane et al. 1992; (6) Guélin et al. 1995

[b] from Anders & Grevesse 1989

[c] average value derived from the above ratios (see text)

[d] due to the non negligeable opacity of the ^{28}Si bearing lines, only a lower limit could be derived.

Molecules as a probe of of magnetic fields

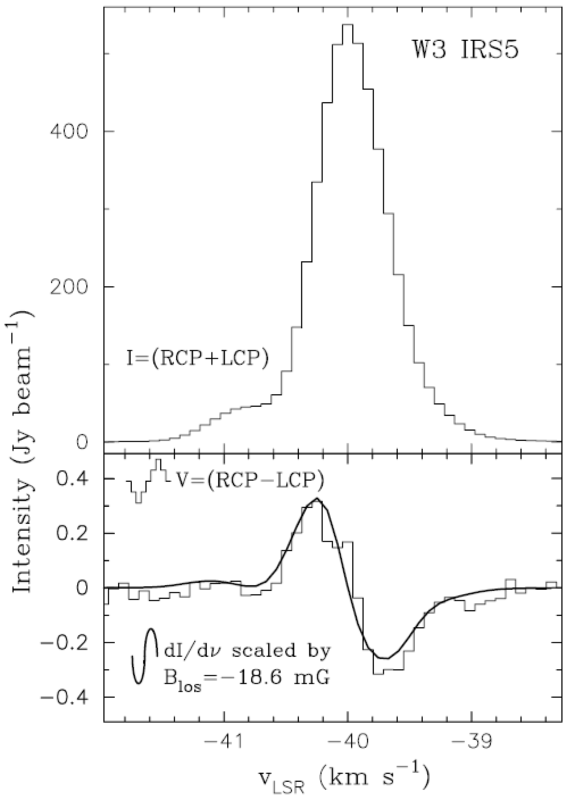


FIG. 2.—Stokes I (*top*) and V (*bottom*) profiles of the −40 km s^{-1} maser in W3 IRS 5 at the position (0, 0) in Fig. 1 (*histograms*). The superposed curve in the bottom panel shows the derivative of I scaled by $B_{los} = -18.6 \pm 1.3$ mG. The V profile is that obtained after leakage correction, i.e., $V - aI$, as described in § 3 (eq. [1]).

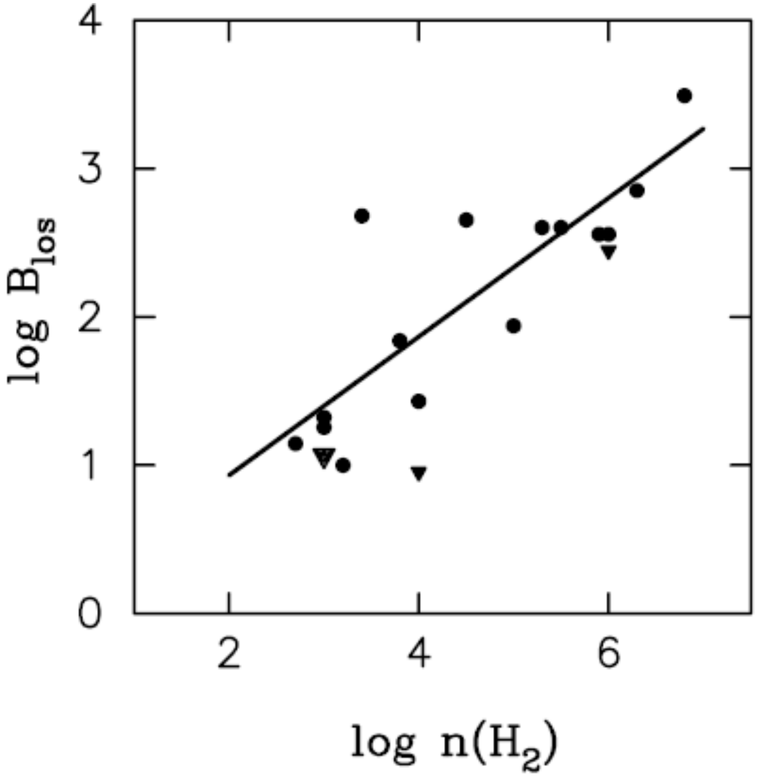


FIG. 1.—Plot of log B_{los} vs. log $n(H_2)$. Inverted triangles are the upper limits for undetected clouds; the averaged limit for all of the dark clouds with log $n(H_2) = 3$ is plotted as a single large inverted triangle. The line is the fit to detected clouds.

Sarma et al. 2002, ApJ

Crutcher 1999, ApJ: $U_{mag} \sim 25\ U_{thermal}$

3) Molecules as coolants

In addition to serving as test particles that can be used as diagnostic probes, molecular emissions can dominate the thermal balance in astrophysical objects

Examples:

- Star-forming molecular clouds
- Primordial gas in the Early Universe

Molecular cooling in the Early Universe

H_2 played a key role as a coolant in the Early Universe before heavy elements had been created (from Lepp & Shull, 1984, ApJ)

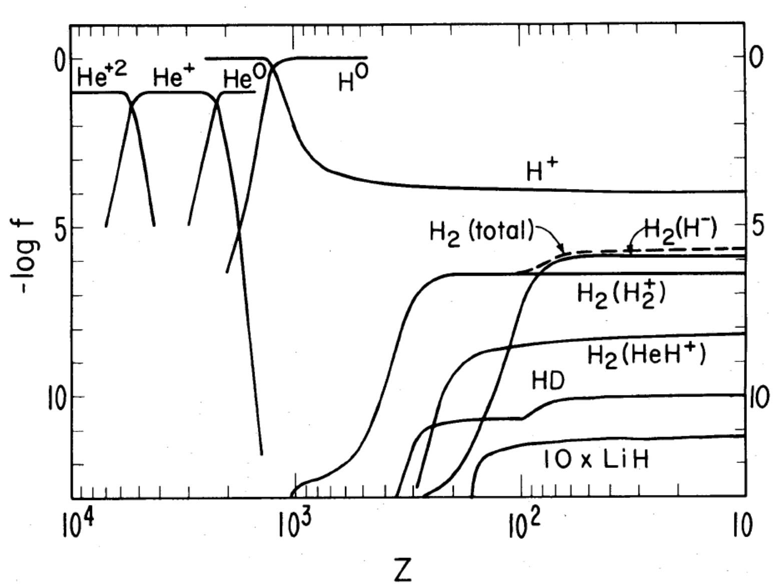


FIG. 1.—Abundance fractions of atoms, ions, and molecules vs. redshift n standard Friedmann cosmological model 1 ($\Omega_0 = \Omega_b = 0.1$, $h = 0.5$; see Table 1). Various channels for H_2 formation are indicated by the intermediate species in parentheses. The HD curve includes both radiative association and the isotropic versions of H_2 formation via H^- and H_2^+.

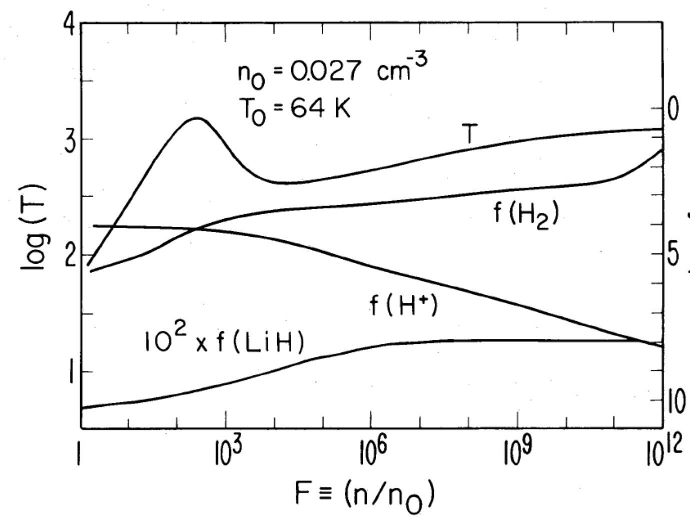


FIG. 3.—Temperature and density track of spherical cloud in self-similar free-fall collapse at redshift $z = 50$. Fractional H^+ and H_2 abundances are plotted at right. Molecular cooling causes temperature to depart from adiabatic value ($T \propto n^{2/3}$) at 10^3 K, and the formation of additional molecules in the collapse cools cloud below 500 K until $n > 10^9$ cm^{-3}, when three-body H_2 formation sets in.

Born-Oppenheimer approximation

- Key realization: to very good approximation, the electron and nuclear motions can be treated separately

  Particle momenta $p \sim \hbar/a_0$ for both nuclei and electrons

  ➔ electron velocities larger by a factor

  m_N/me ~ few x 10^3 – 10^5

  electron kinetic energies larger by the same factor

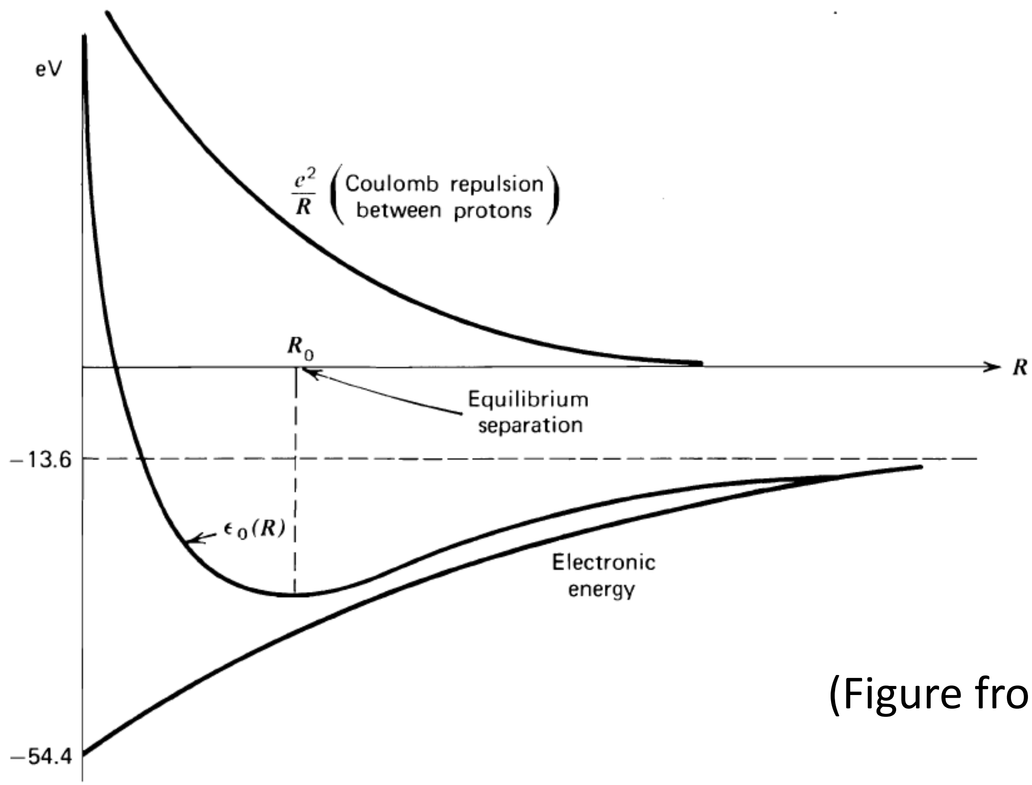
  ➔ initially neglect nuclear k.e. and assume

  that electronic wavefunction and k.e. energy

  responds instantly to slowly changing

  nuclear positions

Potential energy curve



(Figure from Gasiorowicz)

Fig. 20-1. Contributions to "nuclear potential." The Coulomb repulsion and the electronic energy combine to give a curve with a minimum at R_0.

The Born-Oppenheimer approximation underlies the concept of the PES

Different energy scales

- Electronic energy, $E_{el} \sim Ry \sim \hbar 2/m_e a_0{}^2 = \text{few eV}$

- Nuclear vibrational energies $\sim \hbar\omega$

where $\omega \sim \sqrt{}(k/m_N)$ is the classical angular frequency of oscillation for spring constant k

This is related to the curvature of the potential energy curve at its minimum, since

$$E = E_0 + ½\, k(R - R_0)^2$$

Different energy scales

The spring constant, k, must be of order

$$k \sim \frac{E_0}{{R_0}^2} \sim (\hbar^2/mea_0{}^2)\ /a_0{}^2\ = \frac{\hbar^2}{m_e{a_0}^4}$$

Hence $E_{vib} \sim \hbar\sqrt{\frac{\hbar^2}{m_e a^4 m_N}} = \left(\frac{m_e}{m_N}\right)^{\frac{1}{2}}\ Eel \sim \text{few} \times 10^{-2}\ \text{eV}$

- Rotational energies= $\frac{L^2}{2I} \sim \frac{L^2}{m_N\, {R_0}^2} \sim \frac{\hbar^2}{m_N {a_0}^2}$

$$\Rightarrow E_{rot} \sim \left(\frac{m_e}{m_N}\right) E_{el}\ \sim \text{few} \times 10^{-4}\ \text{eV}$$

Summary: $E_{el} : E_{vib} : E_{rot} = 1 : (me/m_N)^{1/2} : (me/m_N)$

Lecture 25
Molecules II

Goals: understand

The LCAO approximation

Molecular orbitals

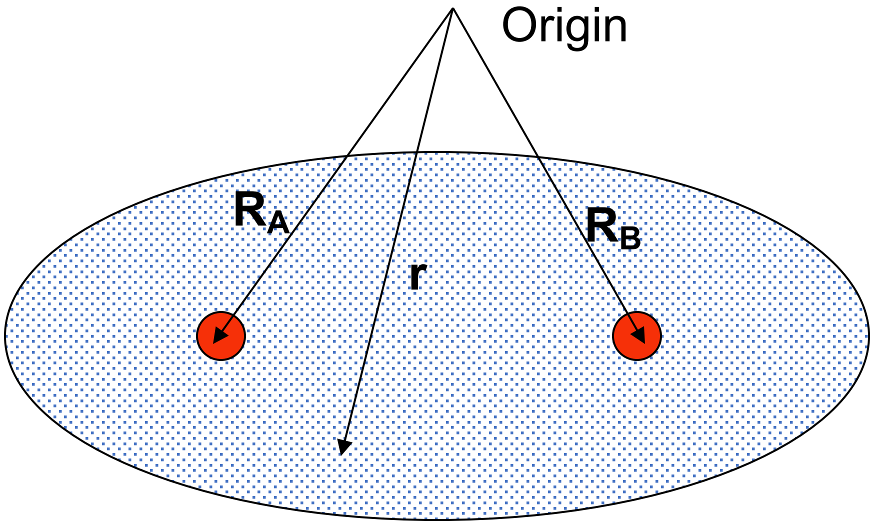
Quantum numbers for molecules

Selection rules

Structure of molecules

Simplest example: H_2^+

(This is analytically tractable)



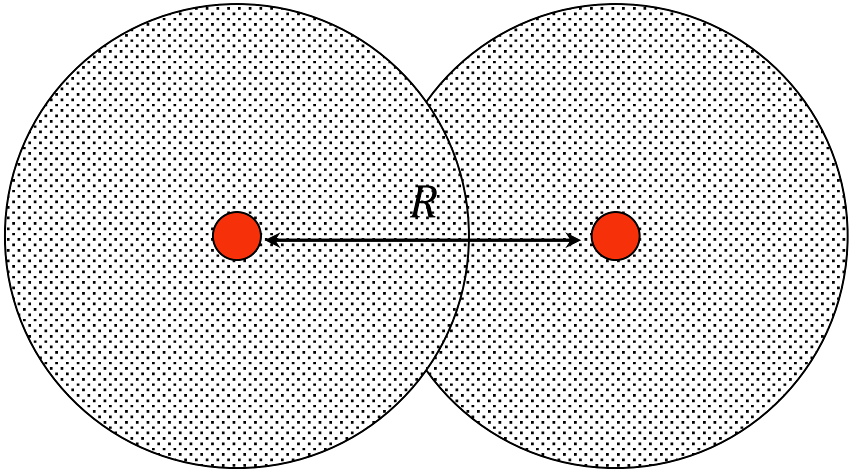
Electronic wavefunction, $\psi(\boldsymbol{r})$ – a function of $R\ =\ |\boldsymbol{R_A} - \ \boldsymbol{RB}|$

Hamiltonian (dimensionless units):

$$H\ = -\frac{1}{2}\nabla^2 - |\boldsymbol{R_A} -\ \boldsymbol{r}|^{\ -1} - |\boldsymbol{R_B} -\ \boldsymbol{r}|^{\ -1} + |\boldsymbol{R_A} -\ \boldsymbol{R_B}|^{\ -1}$$

LCAO – linear combination of atomic orbitals

- The wavefunction of a single electron is called a molecular orbital (MO)
- Approximate this as a linear combination of two atomic 1s orbitals



$\psi(\boldsymbol{r}) \; = \; \alpha\psi_A(\boldsymbol{r}) \; + \; \beta\psi_B(\boldsymbol{r})$ where

$$\psi_A(\boldsymbol{r}) \; = \; \pi^{-1/2}\exp(- \mid \boldsymbol{R_A} - \boldsymbol{r} \mid) \qquad \text{1s orbital}$$
$$\psi_{\boldsymbol{B}}(\boldsymbol{r}) \; = \pi^{-1/2}\exp(- \mid \boldsymbol{R_B} - \boldsymbol{r} \mid)$$

LCAO – linear combination of atomic orbitals

Two linear combinations are consistent with the symmetry of the potential

$\psi_u(\boldsymbol{r}) = Cu\,(\psi_A(\boldsymbol{r}) - \psi_B(\boldsymbol{r}))$ odd parity

$\psi_g(\boldsymbol{r}) = Cg\,(\psi_A(\boldsymbol{r}) + \psi_B(\boldsymbol{r}))$ even parity

with the normalization condition

$1 = \langle \psi_u(\boldsymbol{r}) \,|\psi_u(\boldsymbol{r}) \rangle = \langle\psi_g(\boldsymbol{r}) \,|\psi_g(\boldsymbol{r}) \rangle$

requiring $C_u = (2 - 2S)^{-1/2}$ and $C_g = (2 + 2S)^{-1/2}$

where S = $\langle \psi_A(\boldsymbol{r}) \,|\, \psi_B(\boldsymbol{r}) \rangle$ is the "overlap integral" (which is a function of the internuclear separation, R, of course)

Ritz variational principle

Recall the Ritz variational principle:

If the ground state of any system has energy E_0,

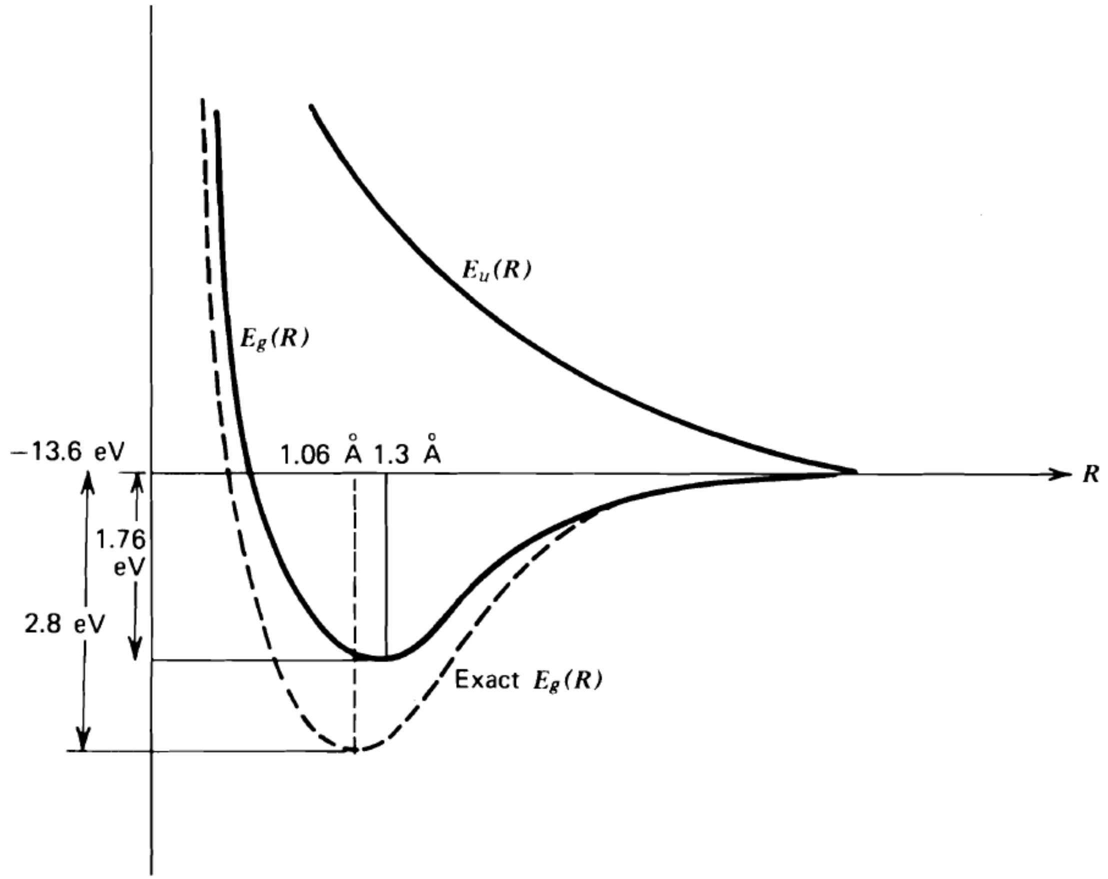
then $\langle \psi \mid H \mid \psi \rangle \geq E_0$ for any function ψ

So E_0 must be smaller than the smaller of $\langle \psi_u \mid H \mid \psi_u \rangle$

and $\langle \psi_g \mid H \mid \psi_g \rangle$

Ritz variational principle

Results of the calculation (figure from Gasiorowicz)



LCAO – linear combination of atomic orbitals

Notes:

1) The even ("gerade") LCAO has the lowest energy:
$\psi_g(\boldsymbol{r}) = Cg\ (\psi_A(\boldsymbol{r}) + \psi_B(\boldsymbol{r}))$

This is called a bonding orbital. The energy is the less than – 13.6 eV except when R is very small

The electron probability density $|\psi_g(\boldsymbol{r})|^2$ peaks in the midplane, where the electron can perform a bonding function

2) The odd ("ungerade") LCAO has an energy that increases monotonically as R decreases

The electron density is zero in the midplane

… an antibonding orbital

LCAO – linear combination of atomic orbitals

Notes:

3) As the internuclear separation R $\rightarrow \infty$, both energies tend to the energy of a hydrogen atom: E = − 13.6eV

The overlap integral tends to 0, and the wavefunctions tend to

$$\psi(\mathbf{r}) = [\psi_A(\mathbf{r}) \pm \psi_B(\mathbf{r})] / \sqrt{2}$$

50% of finding electron on either separated atom

4) As R $\rightarrow$ 0, electronic wavefunction for the exact solution tends to that of an He^+ ion (although our trial solution does not)

$E \rightarrow -54.4\text{eV} + e^2/R$

Nuclear and electronic contributions to the energy
(Figure from Gasiorowicz)

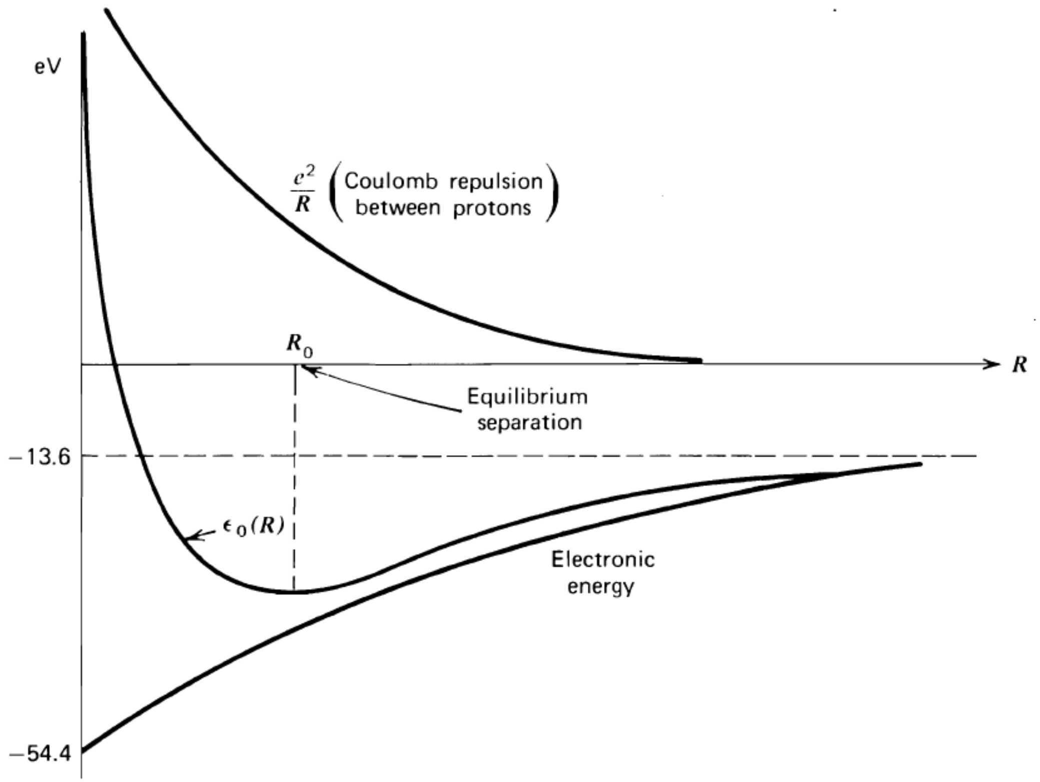


Fig. 20-1. Contributions to "nuclear potential." The Coulomb repulsion and the electronic energy combine to give a curve with a minimum at R_0.

Structure of molecules

Next simplest example: H_2

Two electrons can occupy the same MO.
The Pauli Exclusion Principle requires that they have a wavefunction with an antisymmetric spin part

➔ total spin = 0 (like in the ground state of He).

Structure of molecules

Multielectron diatomic molecules

An electric field points along the internuclear axis and defines a special direction

The projection of angular momenta onto this special axis must be quantized

For an individual electron, m_ℓ takes all integral values between $-\ell$ and ℓ

Define λ = 0, 1, 2, .. l as | m_ℓ |

Multielectron diatomic molecules: orbital angular momentum

For an individual electron, m_ℓ takes all integral values between $-\ell$ and ℓ

Define λ = 0, 1, 2, .. l as $|m_\ell|$

An electron with λ = 0, 1, 2, 3 is called a σ, π, δ, ϕ electron (by analogy with s, p, d, f)

For a collection of electrons, the sum of the λ is denoted Λ. States with Λ = 0, 1, 2, 3 … are denoted Σ, Π, Δ, Φ ..

Structure of molecules

Multielectron diatomic molecules: electronic spin

For a collection of electrons, the total spin is S, and its projection of the onto the internuclear axis is denoted Σ. The spin degeneracy is $2S + 1$, because S takes integer spacing values between $-S$ and S

An electron state can then be characterized as

$^{2S+1}\Lambda$ or $^{2S+1}\Lambda_{(u,g)}$ for a homonuclear molecule

e.g. the ground state of H_2 is a $^1\Sigma_g$ state
the ground state of OH is a $^2\Pi$ state

Structure of molecules

Total electronic angular momentum

The projection of the total electronic angular momentum (spin plus orbital) onto the internuclear axis is $\Omega = \Sigma + \Lambda$

We write this in the form ${}^{2S+1}\Lambda_{\Omega}$

e.g. spin-orbit coupling splits the OH ground state into a ${}^{2}\Pi_{3/2}$ state and a ${}^{2}\Pi_{1/2}$ state

Selection rules

Electronic transitions (dipole rules) :

$\Delta\Lambda = 0, \pm 1$

$\Delta S = 0$

$\Delta\Omega = 0, \pm 1$ but $0 \rightarrow 0$ is forbidden

Homonuclear molecules ($u \leftrightarrow g$ required)

Example: H_2 Lyman ($X^1\Sigma_g \rightarrow B^1\Sigma_u$) and Werner ($X^1\Sigma_g \rightarrow C\ ^1\Pi_u$) bands

Rovibrational splitting

- Perfect harmonic oscillator

$$E_{vib} = \hbar\omega\,(v + ½)$$

v (non-negative integer) is the “vibrational quantum number

- Rigid rotor

$$E_{rot} = (\hbar^2/2I)\,K(K+1) = BK(K+1)$$

Higher order terms arise because of anharmonicity and centrifugal distortion

Rovibrational splitting

Nomenclature: transition from $J_u \rightarrow J_l$ is denoted

$S(J_l)$ for $\Delta J = Ju - J_l = +2$ ← Quadrupole allowed transition

$R(J_l)$ for $\Delta J = Ju - J_l = +1$

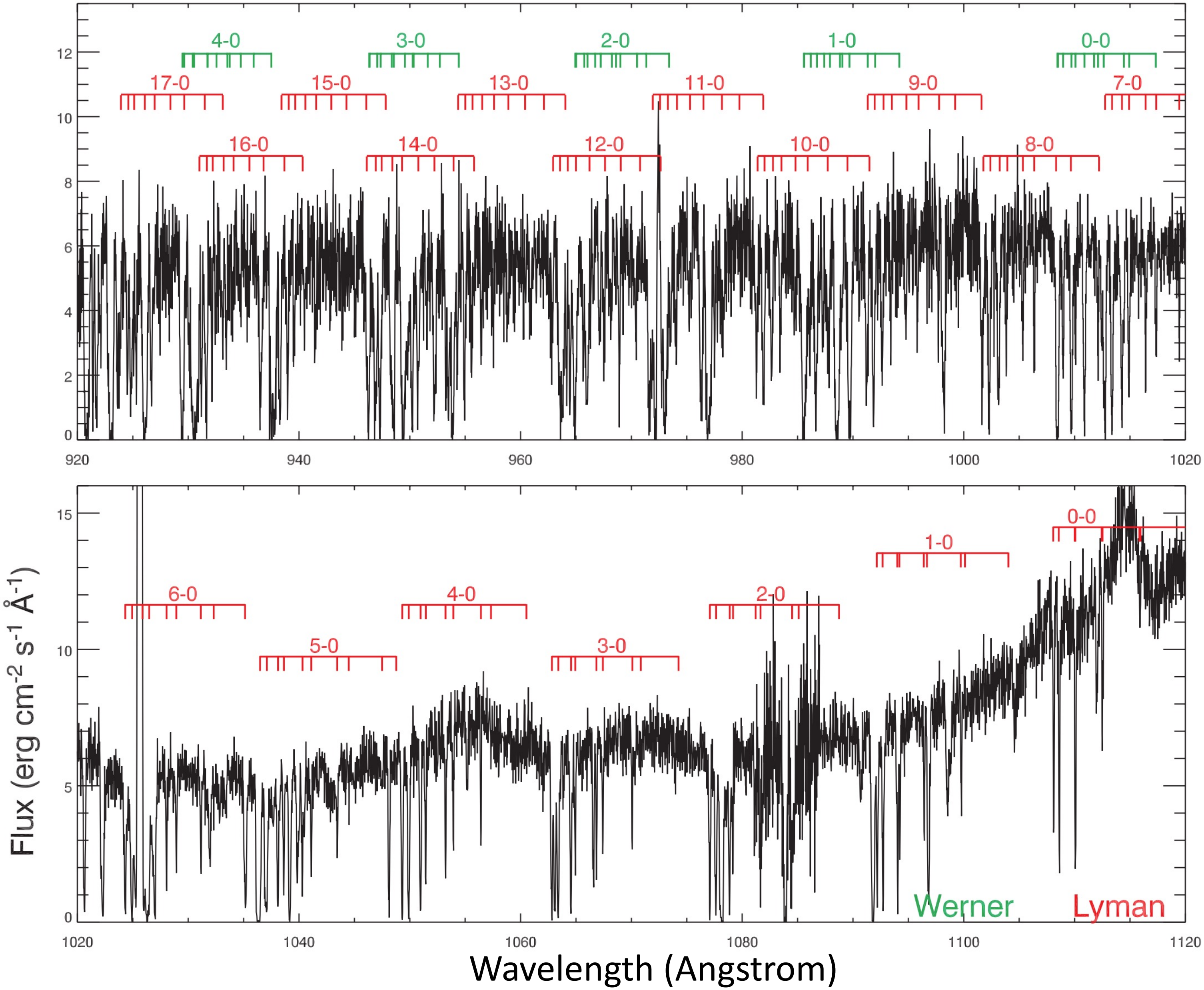
$Q(J_l)$ for $\Delta J = Ju - J_l = +0$

$P(J_l)$ for $\Delta J = Ju - J_l = -1$

Dipole allowed transitions (except that $\Delta J = 0$ is forbidden in $\Sigma \leftrightarrow \Sigma$ transitions)

$O(J_l)$ for $\Delta J = Ju - J_l = -2$ ← Quadrupole allowed transition

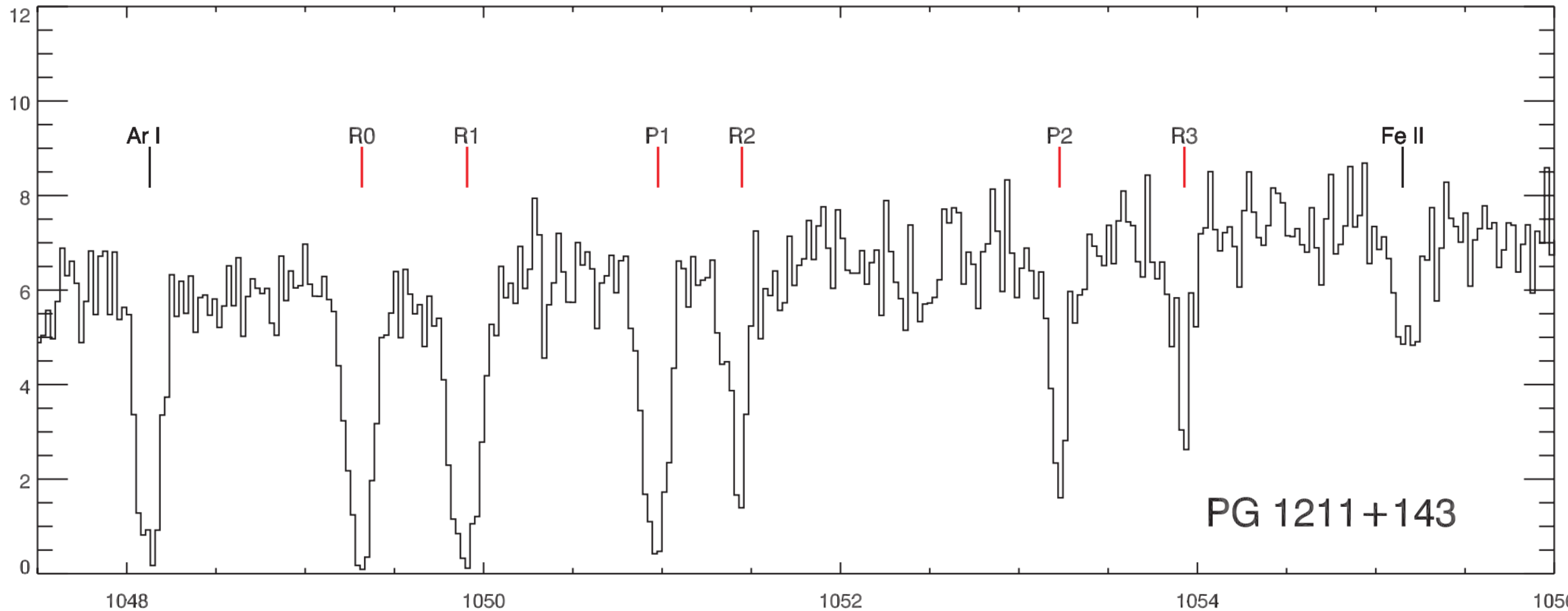
Example: FUV H_2 absorption spectra obtained towards the AGN PG 1211+143 from FUSE



(Gillmon et al. 2006, ApJ)

Example: FUV H_2 absorption spectra obtained towards the AGN PG 1211+143 from FUSE

Zoom in to show rotational structure more clearly



Other splittings, transitions

- Lambda doubling (e.g. OH):
  - For $\Lambda \neq 0$, there are 2 states with a given value of Λ
  - In a non-homonuclear molecule these can have slightly different energies
- Hyperfine splitting (also OH)
- Inversion transitions (e.g. NH_3)
- Torsional and bending modes (e.g. H_2CO)

References on molecular physics

- The classic text: Molecular spectra and molecular structure, by G. Herzberg, (4 volume series)
- Physics and Chemistry of the ISM, by Tielens: pages 21 – 45
- Rybicki and Lightman, Chapter 11
- Journals:
  - Journal of Chemical Physics
  - Journal of Physical Chemistry (A)
  - Molecular Physics
  - Journal of Quantitative Spectroscopy and Radiative Transfer
  - Journal of Physics. B: Atomic, molecular, and optical physics