Quantum Field Theory (171.701), Fall 2022

Problem Set 1

Due: 3 October 2022

1. Show that if A_{ij} is an $N \times N$ matrix and x_i an N-vector, then the "correlation function," defined by,

$$\begin{aligned} &\langle x_i x_j \dots x_k x_l \rangle \\ \equiv \quad & \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_1\, dx_2 \cdots dx_N e^{-\frac{1}{2} x_\alpha A_{\alpha\beta} x_\beta} x_i x_j \dots x_k x_l}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_1\, dx_2 \cdots dx_N e^{-\frac{1}{2} x_\alpha A_{\alpha\beta} x_\beta}}. \end{aligned} \tag{1}$$

Show that

$$\langle x_i x_j \dots x_k x_l \rangle = \sum_{\text{Wick}} (A^{-1})_{ab} \cdots (A^{-1})_{cd}, \tag{2}$$

where "Wick" is a sum over Wick contractions and $\{a, b, \cdots, d\}$ is a permutation of $\{i, j, \cdots, k, l\}$.

2. Work out the propagator $D(x)$ for a scalar free field in (1+1)-dimensional spacetime and study the large-x^1 behavior for $x^0 = 0$.

3. Consider a complex scalar field $\phi(x)$ with lagrangian density,

$$\mathcal{L} = -(\partial^\mu \phi^\dagger)(\partial_\mu \phi) - m^2 \phi^\dagger \phi. \tag{3}$$

(a) Show that ϕ satisfies the Klein-Gordon equation. (b) Treat ϕ and $\phi^\dagger$ as independent fields and find the conjugate momentum for each. Compute the hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives). (c) Write the mode expansion of ϕ as

$$\phi(x) = \int \widetilde{dk} \left[a(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx} \right]. \tag{4}$$

Express $a(\mathbf{k})$ and $b(\mathbf{k})$ in terms of ϕ and $\phi^\dagger$. (d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by $a(\mathbf{k})$ and $b(\mathbf{k})$ and their hermitian conjugates. (e) Express the hamiltonian in terms of $a(\mathbf{k})$ and $b(\mathbf{k})$ and their hermitian conjugates.

4. Write the generating function $Z_o(J, J^\dagger)$ for the complex scalar field with source term $J^\dagger \phi + J \phi^\dagger$ (since the complex scalar field is composed of two real scalar fields, we need two sources to match the two degrees of freedom). Find an explicit formula for $Z_0(J, J^\dagger)$. Use this generating function to calculate the time-ordered correlation functions, $\langle 0 | T\phi(x_1)\phi(x_2) | 0 \rangle$, $\langle 0 | T\phi^\dagger(x_1)\phi(x_2) | 0 \rangle$, and $\langle 0 | T\phi^\dagger(x_1)\phi^\dagger(x_2) | 0 \rangle$.

5. Consider a complex scalar field with $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_0$ is the free-field Lagrangian above, and

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{4} Z_\lambda (\phi^\dagger \phi)^2 + \mathcal{L}_{\rm ct} \\ \mathcal{L}_{\rm ct} &= -(Z_\phi - 1)\partial^\mu \phi^\dagger \partial_\mu \phi - (Z_m - 1) m^2 \phi^\dagger \phi. \end{aligned} \quad (5)$$

   Since this theory has two different types of sources (J and $J^\dagger$), we need to label the type of source on a propagator that points toward the source if it is a $J^\dagger$, and away from the source if it is a J. (a) What kind of vertex appears in the diagrams for this theory, and what is the associated vertex factor (the arrows pointing in and out of that vertex play a role)? (b) Ignoring the counterterms, draw all the connected diagrams with $1 \leq E \leq 4$ and $0 \leq V \leq 2$ and find their symmetry factors.

6. Write down the Feynman rules for the complex scalar field with quartic interactions (from the previous problem). Remember that there are two kinds of particles (which we can think of as positively and negatively charged) and that your rules must have a way of distinguishing them.

7. Consider a theory of two real scalar fields A and B with an interaction $\mathcal{L}_1 = gAB^2$. (a) Assuming that $m_A > 2m_B$, compute the total decay rate of the A particle at tree level. (b) Consider a theory of a real scalar field ϕ and a complex scalar field χ with $\mathcal{L}_1 = g\phi\chi^\dagger\chi$. Assuming that $m_\phi > 2m_\chi$, compute the total decay rate of the ϕ particle at tree level.

8. The Noether current for a theory of two scalar fields ϕ_a (with $a = 1, 2$) with an $O(2)$ symmetry is

$$j^\mu(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi_a(x))} \delta\phi_a(x). \quad (6)$$

   Assuming that the variation $\delta\phi_a$ does not involve time derivatives, use the canonical commutation relations to show that $[\phi_a, Q] = i\delta\phi_a$, where Q is the Noether charge.

9. Consider a theory of N real scalar fields ϕ_i with a Lagrangian density,

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - \frac{1}{2} m^2 \phi_i \phi_i - \frac{\lambda}{16}(\phi_i \phi_i)^2, \quad (7)$$

   that is invariant under an $SO(N)$ transformation,

$$\phi_i(x) \to R_{ij}\phi_j(x), \quad (8)$$

   where R is an orthogonal matrix ($R^T = R^{-1}$) with positive determinant ($\det R = +1$). (a) Find the Noether currents $j^{a\mu}$ for this transformation. (b) Show that $[\phi_i, Q^a] = (T^a)_{ij}\phi_j$, where Q^a are the Noether charges and T^a the generators of the symmetry. (c) Use this result to show that $[Q^a, Q^b] = if^{abc}Q^c$.

10. The elements of the group $SO(N)$ can be defined as $N \times N$ matrices R that satisfy $R_{ii'} R_{jj'} \delta_{i'j'} = \delta_{ij}$. The elements of the symplectic group $\mathrm{Sp}(2N)$ can be defined as $2N \times 2N$ matrices S that satisfy $S_{ii'} S_{jj'} \eta_{i'j'} = \eta_{ij}$, where the symplectic metric η_{ij} is antisymmetric, $\eta_{ij} = -\eta_{ji}$, and squares to minus the identity: $\eta^2 = -I$. One way to write η is

$$\eta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{9}$$

where I is the $N \times N$ identity matrix. Find the number of generators of $\mathrm{Sp}(2N)$.