Quantum Field Theory (171.702), Spring 2023

Problem Set 2

Due: 31 March 2023

1. Consider a theory with a nonabelian gauge symmetry and also a $U(1)$ gauge symmetry. The theory contains left-handed Weyl fields in the representations (R_q, Q_i) where R_i is the representation of the nonabelian group, and Q_i is the $U(1)$ charge. Find the conditions for this theory to be anomaly free.

2. Consider the Lagrangian density for a single massless fermion $\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi$. This Lagrangian is invariant under the $U(1)_V$ transformation $\psi \to e^{i\theta}\psi$ and the $U(1)_A$ transformation $\psi \to e^{i\theta\gamma_5}\psi$, leading to Noether currents J^μ and J^μ_A, the latter of which is violated in the quantum theory; this is the chiral anomaly. The violation of the chiral anomaly can be seen by considering the correlator $\langle 0|T J^\lambda_A(0) J^\mu(x_1) J^\nu(x_2)|0\rangle$, which in Fourier space is represented by two triangle Feynman diagrams with vertices $\gamma^\lambda\gamma^5$ connected to the external momentum $q = k_1 + k_2$, γ^μ connected to an external momentum k_1 and γ^ν connected to k_2. This evaluates to the three-point function,

$$\begin{aligned}\Delta^{\lambda\mu\nu}(q = k_1 + k_2, k_1, k_2) &= -i^3 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\lambda\gamma^5 \frac{1}{\not{p} - \not{q}}\gamma^\nu \frac{1}{\not{p} - \not{k}_1}\gamma^\mu \frac{1}{\not{p}}\right.\\ &\qquad \left. + \gamma^\lambda\gamma^5 \frac{1}{\not{p} - \not{q}}\gamma^\mu \frac{1}{\not{p} - \not{k}_2}\gamma^\nu \frac{1}{\not{p}}\right). \qquad (1)\end{aligned}$$

Now consider the divergence $q_\lambda \Delta^{\lambda\mu\nu}(q, k_1, k_2)$. This has a linear divergence which can be (Pauli-Villars) regulated by replacing $\not{q}\gamma_5$ in the second term by $[2M + (\not{p} - M) - (\not{p} - \not{q} + M)]\gamma_5$. Now shift the integration variable to show that

$$q_\lambda \Delta^{\lambda\mu\nu}(q, k_1, k_2) = -2M\Delta^{\mu\nu}(k_1, k_2), \qquad (2)$$

where

$$\begin{aligned}\Delta^{\mu\nu}(k_1, k_2) &= -i^3 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma^5 \frac{1}{\not{p} - \not{q} - M}\gamma^\nu \frac{1}{\not{p} - \not{k}_1 - M}\gamma^\mu \frac{1}{\not{p} - M}\right.\\ &\qquad \left. + \gamma^\lambda\gamma^5 \frac{1}{\not{p} - \not{q} - M}\gamma^\mu \frac{1}{\not{p} - \not{k}_2 - M}\gamma^\nu \frac{1}{\not{p} - M}\right). \qquad (3)\end{aligned}$$

Show that $\Delta^{\mu\nu}$ goes as $1/M$ in the limit $M \to \infty$, and so $q_\lambda \Delta^{\lambda\mu\nu}(q, k_1, k_2)$ goes to a finite limit as $M \to \infty$.

3. Show by Taylor expanding in powers of $1/g^2$ that the expectation value $\langle W[C] \rangle$ of the Wilson-loop operator for Yang-Mills theory satisfies as an area law; i.e., that this expectation value scales as L^2 for a square contour C with sides of length L.

4. Consider a theory of quarks in which the color group is $SO(3)$ rather than $SU(3)$ and that each quark flavor is represented by a Dirac field in the 3 representation of $SO(3)$. (a) With n_F flavors of massless quarks, what is the nonanomalous flavor symmetry group? (b) Assume the formation of a color-singlet, Lorentz scalar, fermion condensate. Assume that it preserves the largest possible unbroken subgroup of the flavor symmetry. What is this unbroken subgroup? (c) For the case $n_F = 2$, how many massless Goldstone bosons are there? (d) Now suppose that the color group is $SU(2)$ rather than $SU(3)$ and that each quark flavor is represented by a Dirac field in the 2 representation of $SU(2)$. Repeat parts (a), (b), and (c) for this case.