Quantum Field Theory (171.702), Spring 2023

Problem Set 3

Due: 10 May 2023

Problem One: Renormalization of a field theory of fermions in the large-N limit (Problem 17.1 from Fradkin's QFT book.)

Consider the chiral Gross-Neveu model, which is a simple model of chiral symmetry breaking in particle physics, and of charge-density waves in condensed matter. For simplicity, consider the case of $(1+1)$-dimensional spacetime, although it is easy to work out a generalization to higher dimensions. Most of the problems below are formulated for the theory in Minkowski spacetime. Naturally, you will have to rotate the theory to Euclidean spacetime to do the integrals and to derive the RG equations.

The Lagrangian density of the chiral Gross-Neveu model is

$$\mathcal{L} = \bar{\psi}_a i\partial\!\!\!/ \psi_a + \frac{g_0}{2N}\left(\left(\bar{\psi}_a\psi_a\right)^2 - \left(\bar{\psi}_a\gamma_5\psi_a\right)^2\right)$$

where ψ_a is a two-component Dirac spinor

$$\psi_a(x) \equiv \begin{pmatrix} R_a \\ L_a \end{pmatrix}$$

with R_a and L_a being the amplitudes for the (chiral) right and left fields, respectively, with $a = 1, \ldots, N$. In this exercise assume that N is so large that the limit $N \to \infty$ is a reasonable approximation. We will use the basis for the spinors in which the two-dimensional γ-matrices are given in terms of Pauli matrices: $\gamma_0 = \sigma_1, \gamma_1 = i\sigma_2$, and $\gamma_5 = -\sigma_3$, and we use the notation $\not\partial_\mu\gamma^\mu = \gamma_0\partial_0 - \gamma_1\partial_1$. Notice that the usual coupling constant g has been redefined by a scale factor: $g = g_0/2$ N.

1. The Lagrangian of this system contains an interaction term that is quartic in the Fermi fields. Instead of using straightforward perturbation theory you will study this system in the large- N limit. To do this, you first need to verify the following Gaussian identity, also known as a Hubbard-Stratonovich transformation:

$$\int \mathcal{D}\sigma(x) \exp\left(-i\frac{N}{2g_0}\int d^2x\sigma^2(x) - i\int d^2x\sigma(x)\bar{\psi}(x)\psi(x)\right)$$
$$= \mathcal{N}\exp\left(i\frac{g_0}{2N}\int d^2x(\bar{\psi}\psi)^2\right) \tag{1}$$

where $\mathcal{N}$ is a suitable normalization constant, and

$$\bar{\psi}\psi \equiv \sum_{a=1}^{N} \bar{\psi}_a(x)\psi_a(x)$$

The field $\sigma(x)$ does not carry any indices.

2. Use an identity of the type of the one derived in problem 1, involving two scalar fields $\sigma(x)$ and $\omega(x)$, to write the Lagrangian of the chiral Gross-Neveu model in a form that is quadratic in the Fermi fields.

3. This model is invariant under the continuous global chiral transformation $\psi_a = e^{i\theta\gamma_5}\psi'_a$. What transformation law should the scalar fields σ and ω satisfy?

4. Integrate out the Fermi fields, and find the effective action for the scalar fields σ and ω. Watch for the factors of N, and be careful with the signs! By an appropriate rescaling of the scalar fields, show that the effective action has the form $S_{\text{eff}} = N\bar{S}$. Determine the form of $\bar{S}$.

5. Now consider the limit $N \to \infty$. Find the saddle-point equations, which determine the average values of the scalar fields in this limit. Find the solution of the saddle-point equations with lowest energy. Is the solution unique? Use dimensional regularization. What quantities need to be renormalized to make the saddle-point equations finite? How many renormalization constants do you need? Give your answers in terms of coupling-constant and wave-function renormalizations. Be careful to include the dependence on the dimensionality $2+\epsilon$. Determine the renormalization constants using the minimal subtraction scheme.

6. Compute the β function. Find its fixed points and flows in $1+1$ dimensions. Solve the differential equation $\beta(g_0) = \kappa\frac{\partial g_0}{\partial \kappa}$, where κ is a momentum scale. Determine the asymptotic behavior of g_0 in the limit $\kappa \to \infty$. Is the interaction term relevant, irrelevant, or marginal?

7. Use the results of the previous parts to write the saddle-point equations in terms of renormalized quantities alone. In particular, find the dependence of the average values of the scalar fields on the renormalized coupling constant.

8. Consider now the fermion propagator in the $N \to \infty$ limit. Are the fermions massive or massless? If the former is true, what is the value of the fermion mass, and how does it relate to the expectation values of the scalar fields?

9. Find the effective action for the scalar fields to leading order in the $\frac{1}{N}$ expansion (i.e., to order $\frac{1}{N}$). Determine the propagator of the scalar fields at this order. Are the scalar fields massive or massless?

10. Consider now the effect of a field that breaks the chiral symmetry. The extra term in the Lagrangian is $\mathcal{L}_{\text{sources}}$, given by

$$\mathcal{L}_{\text{sources}} = H_0(x)\bar{\psi}_a(x)\psi_a(x) + H_5(x)\bar{\psi}_a(x)\gamma_5\psi_a(x)$$

Find the new effective action of this theory in the presence of these symmetrybreaking fields. Derive the modified saddle-point equations. Solve the new saddle-point equations for the case $H_0(x) = H$ and $H_5(x) = 0$.

11. Repeat the renormalization procedure employed for the theory without sources, now for the case with sources present. Be careful to include a wave-function renormalization. Derive the renormalized saddle-point equations. Renormalize the propagators of part 9 .

12. By functionally differentiating the path integral with respect to the sources, derive an equation of identities that relates expectation values of the scalar fields σ and ω to expectation values of the fermion bilinears $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$. In particular, find a formula that relates the propagators of σ and ω to the propagators of the fermion bilinears.

13. Use the Ward identity you derived in the exercises of chapter 12 to derive a relation between the two-point functions of the scalar fields at zero momentum, and the external symmetry-breaking field. Do the results you found in part 9 satisfy these relations?

14. Derive the RG equation (Callan-Symanzik equation (15.79)) satisfied by the scalar fields in the absence of external sources. Solve these Callan-Symanzik equations in terms of a momentum rescaling factor ρ and a running coupling constant. Note: Unlike renormalized perturbation theory, here you will find a solution of the RG equations that holds for all values of the coupling constant. This is possible because of the large- N limit, which is nonperturbative in the coupling constant.

15. Use the solutions of part 14 to find the asymptotic behavior of the two-point functions of the scalar fields at large momenta.

Problem Two: Fractional charge and statistics in $U(1)_k$ Chern-Simons theory (Fradkin's problem 22.1)

In this exercise consider the $U(1)_k$ abelian Chern-Simons theory with gauge field A_μ coupled to two types of sources, j_μ and an external (background) electromagnetic field A_μ^{em}. The Lagrangian is now

$$\mathcal{L} = \frac{k}{4\pi}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda + j_\mu A^\mu + \frac{e}{2\pi}A_\mu^{\text{em}}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$$

1. Show that the electromagnetic current of this theory is $J_\mu^{\text{em}} = \frac{e}{2\pi}\epsilon_{\mu\nu\lambda}\partial^v A^\lambda$, and show that this current is locally conserved.

2. Consider now the case in which the current j_μ represents the worldlines of very heavy particles that are charged under the Chern-Simons gauge field. Integrate out the Chern-Simons gauge field, and find the effective action for the matter currents j_μ and for the electromagnetic gauge field A_μ^{em}.

3. Use the effective action of the previous part to compute the electromagnetic current induced by the external electromagnetic field. Use the result to compute the conductivity tensor $\sigma_{ij}(i, j = 1, 2)$, such that $J_i^{\text{em}} = \sigma_{ij}E_j^{\text{em}}$, where $\mathbf{E}^{\text{em}}$ is the external electric field.

4. Use the effective action to show that the particles represented by the currents j_μ have electromagnetic charge $q = e/k$ and fractional statistics $\theta = \pi/k$.