Quantum Field Theory (171.701), Fall 2022

Problem Set 2

Due: 7 November 2022

1. Show that the following bilinears in the spinor field $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\sigma^{\mu\nu}\psi$, $\bar{\psi}\gamma^\mu\gamma^5\psi$, and $\bar{\psi}\gamma^5\psi$ transform under the Lorentz group and parity as a scalar, vector, tensor, pseudovector (or axial vector), and pseudoscalar, respectively.

2. Work out the Dirac equation in (1+1)-dimensional spacetime. You should find that the Dirac matrices are 2×2 matrices and that the spinors are two-component objects. Is there a "γ_5" matrix?

3. Work out the Dirac equation in (2+1)-dimensional spacetime. Show that the apparently innocuous mass term violates parity and time reversal. [Hint: The three γ^μ's are just the three Paul matrices with appropriate factors of i.] Note also that there is no γ_5 matrix in (2+1)-d.

4. Use Noether's theorem to derive the conserved current $J^\mu = \bar{\psi}\gamma^\mu\psi$, where $\psi(x)$ is a Dirac spinor. Calculate $[Q, \psi]$ and thus assign the appropriate charges to the creation and annihilation operators in the expansion of $\psi(x)$.

5. Show that a spin-$\frac{3}{2}$ particle can be described by a vector-spinor $\Psi_{a\mu}$, where a is a Dirac-spinor index and μ a vector index. Find the equations of motion (the Rarita-Schwinger equations). [Hint: The object $\Psi_{a\mu}$ has 16 components, which we need to cut down to $2 \times \frac{3}{2} + 1 = 4$ components.]

6. Calculate the tree-level cross section for electron-electron scattering in QED.

7. The charged pion ϕ^- is represented by a complex scalar field φ, the muon μ^- by a Dirac field $\mathcal{M}$, and the muon neutrino ν_μ by a spin-projected Dirac field $P_L\mathcal{N}$, where $P_L = \frac{1}{2}(1 - \gamma_5)$. The charged pion can decay to a muon and a muon antineutrino via the interaction,

$$\mathcal{L}_1 = 2c_1 G_F f_\pi \partial_\mu \bar{\mathcal{M}} \gamma^\mu P_L \mathcal{N} + \text{h.c.}, \qquad (1)$$

where c_1 is the cosine of the Cabibbo angle, G_F is the Fermi constant, and f_π is the pion decay constant. (You may want to verify that f_π has mass dimension 1 and G_F has mass dimension -2.) (a) Compute the charged-pion decay rate Γ. (b) The charged-pion mass is $m_\pi = 139.6$ MeV, the muon mass is $m_\mu = 105.7$ MeV, and the muon neutrino can, for our purposes, be taken to be massless. The Fermi constant is measured in muon decay to be $G_F = 1.166 \times 10^{-5}$ GeV^{-2}, and the cosine of the Cabibbo angle is measured in nuclear beta decays to be $c_1 = 0.974$. The measured value of the charged-pion lifetime is 2.603×10^{-8} sec. Determine the value of f_π in MeV. You should find that your result is larger than the true value by 0.8% because of the neglect of electromagnetic loop corrections.

8. Prove Furry's theorem: Any QED scattering amplitude with no external fermions, and an odd number of external photons, is zero.

9. Show that there are six one-loop diagrams with four external photons (they all look the same; its just a question of permutations). Show that even though each diagram is logarithmically divergent, the sum is finite. Explain why this cancellation is required by gauge-invariance. (By the way, if you evaluate these loops in the limit that the external-photon momenta are small, you will obtain the Euler-Heisenberg Lagrangian, the first non-vanishing correction to the gauge-field Lagrangian $\mathcal{L} = (1/4)F^{\mu\nu}F_{\mu\nu}$.)

10. Consider the current-current correlation function $\langle 0|Tj^\mu(x)j^\nu(y)|0\rangle$ in QED.
(a) Show that its Fourier transform is proportional to

$$\Pi^{\mu\nu}(k) + \Pi^{\mu\rho}(k)\tilde{\Delta}_{\rho\sigma}(k)\Pi^{\sigma\nu}(k) + \cdots, \tag{2}$$

where $\Pi^{\mu\nu}(k)$ is the photon self energy and $\tilde{\Delta}_{\rho\sigma}(k)$ the photon propagator.
(b) Use this result to prove that $\Pi^{\mu\nu}(k)$ is transverse; i.e., that $k_\mu\Pi^{\mu\nu}(k) = 0$.¡