Quantum Field Theory (171.701), Fall 2022

Problem Set 2

Due: 7 November 2022

1. Consider the Landau-Ginsberg free energy for a complex scalar ψ,

$$F = \int dx \left(\alpha_2(T)|\psi|^2 + \alpha_4|\psi|^4 - \gamma \left|\frac{d\psi}{dx}\right|^2 + \kappa \left|\frac{d^2\psi}{dx^2}\right|^2 \right), \tag{1}$$

with $\gamma, \alpha_4, \kappa > 0$. Consider an ansatz in which a single Fourier mode $\psi_k = A_k e^{ikx}$ with $k = \pm k_0$ is nonvanishing. What value of k_0 minimizes the free energy? Show that the system undergoes a phase transition to a spatially modulated phase when $\alpha_2 = \gamma^2/4\kappa$. What symmetries are broken in the ordered phase?

2. Consider the free energy for a complex scalar field ψ coupled to a gauge field A_i in d dimensions,

$$F[\psi, A_i] = \int d^d x \left[\frac{1}{4} F_{ij}F^{ij} + |\partial_i\psi - ieA_i\psi|^2 + \mu^2|\psi|^2 \right], \tag{2}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$. (As an aside: if we add a quartic term $g|\psi|^4$, then, in $d = 3$, this is the original Ginzburg-Landau free energy for a superconductor.) What is the critical dimension d_c, such that the coupling between the scalar and gauge field is relevant for $d < d_c$ and irrelevant for $d > d_c$? Speculate on what might happen at $d = d_C$.

3. The $O(N)$ mode consists of N scalar fields $\phi = (\phi_1, \cdots, \phi_N)$, with a free energy,

$$F_1[\phi] = \int d^d x \left[\frac{1}{2} (\nabla\phi)^2 + \frac{1}{2}\mu^2 (\phi\cdot\phi) + g (\phi\cdot\phi)^2 \right], \tag{3}$$

which is invariant under $O(N)$ rotations. In $d = 4 - \epsilon$ dimensions, the beta functions become,

$$\begin{aligned} \frac{d\mu^2}{ds} &= 2\mu^2 + \frac{N+2}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+\mu^2}\tilde{g}, \\ \frac{d\tilde{g}}{ds} &= \epsilon\tilde{g} - \frac{N+8}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+\mu^2)^2}\tilde{g}^2, \end{aligned} \tag{4}$$

where $\tilde{g} = \Lambda^{-\epsilon} g$ is the dimensionless coupling constant. What is the critical exponent ν at the Wilson-Fisher fixed point. Assuming that $\eta \sim \mathcal{O}(\epsilon^2)$, determine the critical exponents α, β, γ, and δ to leading order in ϵ.

4. The free energy of the Sine-Gordon model in $d = 2$ dimensions, with UV cutoff Λ, is given by

$$F[\phi] = \int d^2x \left[\frac{1}{2}(\nabla\phi)^2 - \lambda_0 \cos(\beta_0\phi)\right], \tag{5}$$

What is the "naive" dimension of β_0 and λ_0. Decompose the modes of the field as $\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^- + \phi_{\mathbf{k}}^+$, where ϕ_k^+ is nonzero for $Lambda/\zeta < k < \Lambda$, and ϕ_k^- nonzero for other k. Let $\phi^-(\mathbf{x})$ and $\phi^+(\mathbf{x})$ be the inverse Fourier transforms. Show that after integrating out $\phi_{\mathbf{k}}^+$, the free energy for ϕ^- becomes, to leading order in λ_0,

$$F'[\phi^-] = \int d^2x \left[\frac{1}{2}(\nabla\phi^-)^2 - \lambda_0 \left\langle\cos\beta_0(\phi^- + \phi^+)\right\rangle_+\right], \tag{6}$$

where you should define the meaning of $\langle\,\rangle_+$. Evaluate this expectation value to show that

$$F'[\phi^-] = \int d^2x \left[\frac{1}{2}(\nabla\phi^-)^2 - \lambda_0 \zeta^{-\beta_0^2/4\pi} \cos(\beta_0\phi^-)\right]. \tag{7}$$

Hence, show that the $\cos(\beta\phi)$ potential is relevant when $\beta_0^2 < 8\pi$ and irrelevant when $\beta_0^2 > 8\pi$. [Hint: Write $\cos(\phi^- + \phi^+) = \frac{1}{2}(e^{i\phi^-}e^{i\phi^+} + e^{-i\phi^-}e^{-i\phi^+})$ and use Wick's identity,

$$\left\langle e^{B_a\phi_a}\right\rangle = e^{\frac{1}{2}B_a\langle\phi_a\phi_b\rangle B_b}, \tag{8}$$

for any constant B_a. You will also need the position-space correlation function,

$$\left\langle\phi^+(\mathbf{x})\phi^+(\mathbf{y})\right\rangle_+ = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}. \tag{9}$$

You should make use of this in the limit $\mathbf{x} \to \mathbf{y}$.]

5. Consider a field φ_i in a representation R_1 of some gropu and a field χ_I in a representation R_2. Their product $\phi_i\chi_I$ is then in the direct product representation $R_1 \otimes R_2$, with generator matrices given by

$$\left(T^a_{R_1\otimes R_2}\right)_{iI,jJ} = \left(T^a_{R_1}\right)_{ij}\delta_{IJ} + \delta_{ij}\left(T^a_{R_2}\right)_{IJ}. \tag{10}$$

(a) Prove the distribution rule for the covariant derivative,

$$\left[D_\mu(\varphi\chi)\right]_{iI} = \left(D_\mu\varphi\right)_i\chi_I + \varphi_i(D_\mu\chi)_I. \tag{11}$$

(b) Consider a field φ_i in a representation and its Hermitian conjugate $\varphi_i^\dagger$ in the complex-conjugate representation . Show that

$$\partial_\mu(\varphi^{\dagger i}\varphi_i) = (D_\mu\varphi^\dagger)^i\varphi_i + \varphi^{\dagger i}(D_\mu\varphi). \tag{12}$$

Explain why this is a special case of equation (11), and also why it makes sense.