Ay123

Fall 2005

STELLAR STRUCTURE AND EVOLUTION

Problem Set 2

Due in class, Tuesday, October 11, 2005

1. In the last problem set, you considered a star with a linear density profile. A slightly more realistic stellar model can be obtained by solving the equations of hydrostatic equilibrium for a self-gravitating sphere of gas with a pressure-density relation, $p \propto \rho^\gamma$ [these are called polytropes of index $n = (\gamma - 1)^{-1}$]. The value $\gamma = 5/3$ is a good approximation for low-mass stars ($M < 0.3\, M_\odot$), and $\gamma = 4/3$ is a good approximation for high-mass stars ($M > 50\, M_\odot$), as we will see later. As we will also see later that

$$\gamma = 5/3 \quad \rho_c/\bar{\rho} = 5.99 \quad p_c/(GM^2/R^4) = 0.77 \quad \Omega/(-GM^2/R) = 6/7,$$

$$\gamma = 4/3 \quad \rho_c/\bar{\rho} = 54.18 \quad p_c/(GM^2/R^4) = 11.05 \quad \Omega/(-GM^2/R) = 3/2,$$

where $\bar{\rho} = 3M/(4\pi R^3)$ is the mean density, the subscripts 'c' denotes the central value, and Ω is the gravitational binding energy. Note that the dimensionless numbers are very different from each other, and from the uniform-density (Hansen-Kawaler-Trimble, section 1.4) and linear (problem 1 from last homework) model values. Can you explain crudely why?

   a. Assume the matter in the stars is 75% H by mass and 25% He. Calculate the mean mass μ per particle and the mean mass μ_e per electron (assuming the gas is fully ionized).

   b. If the pressure at the center of a low-mass star (approximated by a $\gamma = 5/3$ polytrope) is all from a monatomic, nonrelativistic ideal gas, show that $T_c = 0.539\, GM\mu m_u/(kR)$, where m_u is the atomic mass unit, and k the Boltzmann constant.

   c. For low-mass stars it is observed (and appropriate variations of the homology relations we derived in class can sort of explain) that stars of mass M have radii $R = R_\odot(M/M_\odot)^0.94$. Assume that the density profile of these stars is about that of a $\gamma = 5/3$ polytrope. Use the observational mass-radius relation to calculate, as a function of $M/M_\odot$: ρ_c (in g/cm^3), p_c (in dyne/cm^2), and T_c (in K). In deriving T_c assume that the pressure is purely ideal-gas thermal pressure of nondegenerate ions and electrons. Compare numbers from your formulae to the "real-model" values tabulated in Table 2.6 of Hansen-Kawaler-Trimble and find where this approximation seems to be good.

d. For what mass stars will the central radiation pressure $aT_c^4/3$ become equal to p_c? Are more massive stars held up mainly by gas pressure or by radiation pressure?

e. The very first stars to form in the Universe are believed to be very massive, and have $R = 5.5R_\odot(M/100\,M_\odot)^{0.45}$. Assume that the density profile of these stars is described by a $\gamma = 4/3$ polytrope. Use this theoretical result to calculate, as a function of $M/100\,M_\odot$: ρ_c (in g/cm^3), p_c (in dyne/cm^2), and T_c (in K). In deriving T_c assume that the central pressure is mainly radiation pressure.

f. Below what mass will primordial stars be supported mainly by gas pressure rather than radiation pressure? Compare with your results in part (d).

2. The opacity in stellar interiors.

a. Integrate the equation of radiative equilibrium, assuming a Rosseland mean opacity law of the form $\kappa = \kappa_0 \rho T^{-s}$ (e.g., $s = 3.5$ for Kramer's law) to find the relation between temperature and pressure as a function of depth in a star in radiative equilibrium. In doing so, assume that radiation pressure may be neglected compared to gas pressure.

b. Show that the surface temperature and pressure boundary conditions become unimportant to the pressure-temperature relation in the deep interior (i.e., all solutions approach a "radiative zero solution").

c. Show that if $s = 3$ that the pressure-temperature (and hence density-pressure) relation is exactly that of a $\gamma = 4/3$ polytrope like that of the previous problem, and if $s = 3.5$ it is nearly the same.

d. Using the relations for central pressure and density in the previous problem, show that the opacity constant of a radiative star is given by (in cgs units)

$$\kappa_0 = 2.8 \times 10^{19} \frac{(M/M_\odot)^5(\mu/0.6)^7}{(L/L_\odot)} \quad \text{for } s = 3,$$

and

$$\kappa_0 = 9.5 \times 10^{22} \frac{(M/M_\odot)^5(\mu/0.6)^7}{(L/L_\odot)} \quad \text{for } s = 3,$$

e. Since for a given chemical abundance κ_0 is a physical constant, your results in (c) give a mass-luminosity-radius relation. Explain why this is independent of radius for $s = 3$.

f. See if you get the same values of κ_0 for the model stars in Table 2.5 of Hansen-Kawaler-Trimble. In what regions does it begin to vary significantly? and why? What about for stars which have evolved off the main sequence (Fig. 2.5 of Hansen-Kawaler-Trimble)?

3. Consider a family of stars in which the opacity is dominated by Thomson scattering by electrons, and in which the nuclear energy is generated by the carbon-nitrogen cycle.

This implies that the opacity is independent of the density and temperature [see Eq. (5.14) in Hansen-Kawaler-Trimble], and that the rate of nuclear energy production is proportional to $\rho^2 T^{18}$. In analogy with the homology relations we derived in class, find for this family of stars a relation between the radius and the mass, and a relation between the luminosity and the mass. Find also the line on the Hertzprung-Russell diagram describing the luminosity and effective surface temperature for these stars.

4. Do problem 5 in Chapter 2 of Hansen-Kawaler-Trimble. (You can skip part 5.)