Ay123

Fall 2005

STELLAR STRUCTURE AND EVOLUTION

Problem Set 3

Due Tuesday, October 18, 2003

1. Do Exercise 1 in Chapter 3 of Hansen-Kawaler-Trimble. (You should be able to explain or derive the degeneracy factors for the ground-state helium atoms and ions, rather than copy them out of Allen.)

2. The electron-positron pair instability. Begin by recalling that since electrons and positrons can annihilate to two photons, $e^+e^- \leftrightarrow \gamma\gamma$, and since photons have chemical potential $\mu_\gamma = 0$, it follows that the electron-positron chemical potentials satisfy $\mu_+ = -\mu_- \equiv -\mu$. Also recall that if there is an excess of electrons over positrons (as may be required to balance the positive charge of nuclei in the plasma), then

$$\frac{\rho}{\mu_e m_u} = n_{e^-} - n_{e^+} = \frac{8\pi}{h^3}\left[\int_0^\infty \frac{p^2\,dp}{e^{(E-\mu)/kT}+1} - \int_0^\infty \frac{p^2\,dp}{e^{(E+\mu+2m_e c^2)/kT}+1}\right],$$

where

$$E = m_e c^2\left[\sqrt{(p/m_e c)^2+1} - 1\right]$$

is the electron kinetic energy, and ρ is the mass density, μ_e is the mean gas mass per free electron, and m_u the atomic mass unit.

   a. Show that as $\rho \to 0$, charge neutrality requires $\mu = -m_e c^2$, while for $\rho > 0$, $\mu > -m_e c^2$.

   b. Give integral expressions for the pressure P and energy density U analogous to those for n_{e^-} and n_{e^+} given above.

   c. Evaluate your expressions for ρ and U in the ultrarelativistic limit $kT \gg m_e c^2$ for $\rho \to 0$ (i.e., for a pure pair plasma) and compare to the analogous expression for photons.

   d. Now consider a gas composed of photons and pairs in equilibrium, again for $\rho \to 0$. Show that at low temperature, $P \simeq aT^4/3$ and that the entropy per baryon is $s \simeq (4/3)aT^3/\rho$. This latter result follows from the first law of thermodynamics once you have obtained $U = 3P$.

   e. Now show that at high temperatures ($kT \gg m_e c^2$), $P = (11/12)aT^4$ and $s = (11/3)aT^3/\rho$.

f. Combine your results to show that at low temperatures $P = 0.23(\rho s)^{4/3}/a^{1/3}$ while at high temperatures $P = 0.16(\rho s)^{4/3}/a^{1/3}$. Sketch an adiabat on the P-ρ plane. Use the sketch to show that for $kT \sim m_e c^2$, it must be that $\Gamma_1 \equiv (\partial \ln P/\partial \ln \rho)_s < 4/3$.

g. Discuss the stability of a star whose central temperature reaches this range.

h. Using the techniques for manipulating thermodynamic derivatives discussed in Section 3.7 of Hansen-Kawaler-Trimble, show that for a gas of electron-positron pairs and radiation with $\beta \equiv kT/m_e c^2 \ll 1$,

$$\Gamma_1 \simeq \frac{4}{3} - \frac{5}{8}\left(\frac{2}{\pi}\right)^{7/2} \frac{e^{-1/\beta}}{\beta^{7/2}}.$$

You may find it handy to begin with $\Gamma_1 = (c_P/c_{V_\rho})\chi_\rho$, $c_{V_\rho} = (\partial E/\partial T)_\rho$ and $c_P - c_{V_\rho} = (P/\rho T)\chi_T^2/\chi_\rho$. You may also want to first derive the electron-positron number density,

$$2n_+ = 2n_- = 4\left(\frac{m_e c}{h}\right)^3 \left(\frac{2\pi kT}{m_e c^2}\right)^{3/2} e^{-m_e c^2/kT} \equiv f(T),$$

and also the energy per unit mass,

$$E = \frac{f(T)}{\rho}\left(m_e c^2 + \frac{3}{2}kT\right) + aT^4/\rho,$$

and pressure

$$P = aT^4/3 + f(T)kT + \rho kT/\mu.$$

You will also need to note at some point that the gas pressure and density are small (although you need to keep track of them to calculate appropriate derivatives). You may also need to note that

$$Tf'(T) \simeq \frac{m_e c^2}{kT} f(T) \gg f(T),$$

for $kT \ll m_e c^2$.

i. Show that additional electrons from ions will increase Γ_1 above the pure-pair value you derived above. Thus at each temperature there will be a critical baryon density above which Γ_1 will drop below the stability limit of 4/3. Roughly estimate that density (for pure hydrogen) at $T \sim 3 \times 10^9$ K. [Hint: The discussion around Eqs. (3.104)–(3.112) in Hansen-Kawaler-Trimble might be useful.]

3. In this problem you will numerically integrate the equation of hydrostatic equilibrium to obtain the mass-radius relation for white dwarfs. You will assume that the pressure in the white dwarf is provided entirely by electron degeneracy pressure, Eq. (3.53) in Hansen-Kawaler-Trimble.

a. Show that

$$\frac{dP}{d\rho} = \frac{m_e c^2}{3\mu_e m_u} \frac{x_F^2}{(1+x_F^2)^{1/2}}.$$

b. Now devise a numerical scheme to integrate the two coupled differential equations,

$$\frac{d\rho}{dr}\frac{d\rho}{dP}\frac{dP}{dr}, \qquad \text{and} \qquad \frac{dm(r)}{dr} = 4\pi r^2 \rho(r),$$

using the equation of hydrostatic equilibrium for dP/dr. You can use Mathematica, write a C or Fortran program, or any other numerical procedure you find handy.

b. As you change the central density, you should get a one-parameter family of stars with a sequence of masses M and radii R. You should be able to understand the qualitative behavior of your results for $R \to 0$ and $R \to \infty$.

c. Plot this mass-radius relation for $\mu_e = 56/26$ for an iron white dwarf and $\mu_e = 2$ for a carbon white dwarf. Plot the following three white dwarfs on your graph: (1) Sirius B, $M = 1.053\, M_\odot$, $R = 0.0074\, R_\odot$; (2) 40 Eri B, $M = 0.48\, M_\odot$, $R = 0.0124\, R_\odot$; (3) Stein 2051, $M = 0.50\, M_\odot$ or $0.72\, M_\odot$, $R = 0.0115\, R_\odot$, and try to infer their compositions.

d. The results you have derived above should show that as $M \to 0$, $R \to \infty$. Clearly, at some point this result must break down (think about where Jupiter would fall on this plot!). This is because when the density becomes sufficiently low, Coulomb interactions between the electrons and ions (neglected above) become important in determining the equation of state. Estimate the Coulomb energy per electron. Use this result to estimate the density at which Coulomb effects become important. You can then estimate (very roughly) the maximum radius for a white dwarf and the mass at which it occurs.

4. In this problem you will consider the opacity due to the negative hydrogen ion H^- in stellar atmospheres. In cool stars, bound-free (photoionization) of this ion provides most of the opacity at wavelengths above the Balmer limit ($\lambda = 3647\, \mathring{A}$, below which opacity is due to photoionization of hydrogen atoms in the $n = 2$ level). The ionization energy of H^- is $\chi_- = 0.754$ eV, meaning that this ion can absorb photons with wavelengths $\lambda < 16,444\, \mathring{A}$.

   a. First, use the Saha equation to write an expression for $n(H^-)/n(H)$, the ratio of the hydrogen-ion to neutral-hydrogen abundances, in terms of χ_-, the temperature T, and the electron pressure, $P_e = n_e kT$ (where n_e is the free-electron number density; these free electrons are provided in cool stellar atmospheres presumably by metals, which have lower ionization thresholds than hydrogen or helium).

   b. Next, calculate the ratio $n(H; n = 2)/n(H; n = 1)$ of the abundance of hydrogen atoms in the first excited state ($n = 2$) to the abundance in the ground state ($n = 1$); then do the same for $n(H; n = 3)/n(H; n = 1)$.

c. Assuming an electron pressure, $P_e = 10^{1.5}$ dyne/cm^2, at what temperature does is $n(H; n = 3) \simeq n(H^-)$? Assuming the bound-free absorption coefficients for H^- and the $n = 3$ level of hydrogen are comparable, this gives the temperature below which H^- aborption dominates the opacity above the Balmer break.

d. Write an expression for $\kappa(3647^+)/\kappa(3647^-)$, the ratio of the opacities just above and below the Balmer break for (i) low-temperature atmospheres where the opacity above the Balmer break is due to the H^- ion, and (ii) high-temperature atmospheres where the opacity above the Balmer break is due to the $n = 3$ level of neutral hydrogen. Your results should show that at low temperatures, the Balmer discontinuity depends on both temperature and electron pressure, while at high temperature, it depends only on temperature.

e. *Extra credit:* If you are so inclined, look up values for the bound-free absorption coefficients for H^- and the $n = 2$ and $n = 3$ levels of hydrogen to fill in numerical values for the ratio in part (d).