Ay123

Fall 2005

STELLAR STRUCTURE AND EVOLUTION

Problem Set 7

Due Tuesday, November 15, 2005

1. Show that the Hayashi line in the HR diagram for a given mass moves to the right (lower temperatures) as the polytropic index n is decreased. This therefore demonstrates that there should be no stars to the right of the Hayashi line in the HR diagram.

2. This is a numerical exercise that deals with adiabatic radial pulsations of a polytrope. You will calculate numerically and plot the lowest eigenfunctions for adiabatic radial perturbations for a polytrope of index $n = 5/3$.
A polytrope of index n represents a state of hyrostatic equilibrium for a star with an equation of state $P \propto \rho^{1+1/n}$. Suppose we perturb the star by moving each mass element at radius r in the star in the radial direction by a distance $\xi(r)$. Normal modes of this perturbation will oscillate as $e^{i\omega t}$ with time with angular frequency ω; if ω is imaginary, the normal mode is unstable and its amplitude grows exponentially. The frequencies ω and normal modes $\xi(r)$ are found simultaneously from the eigenvalue equation,
$$\frac{d}{dr}\left(\Gamma_1 P \frac{1}{r^2}\frac{d}{dr}(r^2\xi)\right) - \frac{4}{r}\frac{dP}{dr}\xi + \omega^2\rho\xi = 0.$$
This equation is derived and discussed in HW Eq. (10.16) and in KW Eq. (38.15), although with different notation. The boundary conditions for the equation are $\xi(r = 0) = 0$ (i.e., mass at the center does not move) and $\xi(r = R)$ =finite at the surface of the unperturbed star. This eigenvalue equation is an example of a Sturm-Liouville equation meaning that (i), there are an infinite set of eigenvalues ω^2, with $\omega_0^2 < \omega_1^2 < \omega_3^2 \cdots$; (ii) the eigenfunction ξ_0 corresponding to ω_0^2 has no nodes in the interval $0 < r < R$; more generally $\xi(n)$ has n nodes in this interval. The eigenfunctions also form a complete orthornormal set.
Section 38.2 in KW solves for the lowest two eigenvalues and eigenfunctions for the $n = 0$ (homogeneous sphere) polytrope. Your assignment is to calculate numerically the lowest three eigenvalues and eigenfunctions for the $n = 1.5$ polytrope with $\Gamma_1 = 5/3$. You can do so any way you like; Ch. 7 in HKT may provide some useful suggestions, and so may KW Section 38.3. The problem is probably most easily done with Mathematica, but there will probably not be a canned routine for it. Here are some hints: (i) Non-dimensionalize the eigenvalue equation; (ii) Determine the radial density/pressure profile

for the nonperturbed polytrope, as you did in the previous homework; (iii) Investigate analytically the behavior of the solution near $r = 0$ and $r = R$; (iv) Your best bet may be to use a shooting method. To do so, you may first integrate out from $r = 0$ and in from $r = R$ to some convenient r inside the star with a guessed value of ω^2, imposing the boundary conditions in step (iii). Since you do not know the relative magnitudes of these two solutions, start them both with unit magnitude. If you had guessed the correct value of ω^2, the Wronskian of the two solutions at the "matching radius" r would vanish (you should understand why). In general, the Wronskian will be nonzero. You can then choose another value of ω^2, integrate it from the boundaries to r and compute the Wronskian again. Now interpolate (or extrapolate) to that value of ω^2 that makes the Wronskian vanish. Continue to iterate until ω^2 converges to a satisfactory number of significant digits.

Plot your results for the lowest eigenfunctions. They should look similar (but not exactly) like the results for $n = 3$ shown in KW Fig. 38.2, or $n = 2$ shown in HKT Fig. 8.1. Compare your results with the $n = 0$ case plotted in KW Fig. 38.1, and give a simple physical argument as to why they look different.