General Relativity Ph236b
Problem Set 10
Due: To Jonathon or his mailbox, March 14, 2005

Suggested Reading: Carroll, Sections 1.10, 4.3, 4.8, and Chapter 9.

1. Problem 4.2 in Carroll.

2. Show explicitly that the spherically symmetric static vacuum spacetime that minimizes the Einstein-Hilbert action is the Schwarzchild metric. To do so, write the metric in the form,
$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\,d\Omega^2,$$
and show explicitly that the Einstein-Hilbert action is invariant to any linear variations to $A(r) = (1 - 2M/r)$ and $B(r) = (1 - 2M/r)^{-1}$.

3. In this problem you will calculate the parameterized post-Newtonian (PPN) parameter γ in Brans-Dicke theory in terms of the Brans-Dicke parameter ω, and you will also relate the Newtonian gravitational constant to ω and the value of the Brans-Dicke scalar λ (in Carroll's notation). To do so, consider the weak-field limit of the spherically-symmetric static vacuum spacetime that surrounds a spherical massive body (e.g., the Sun). Thus, write the spacetime metric as
$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\,d\Omega^2,$$
with $A(r) = 1 + a(r)$ and $B(r) = 1 + b(r)$, where $a(r) \ll 1$ and $b(r) \ll 1$ (and are both linear in the central mass M) in the weak-field limit. You will also need to write the Brans-Dicke scalar as $\lambda = \lambda_0[1 + \epsilon(r)]$, where λ_0 is the value of the Brans-Dicke scalar at large distances, and $\epsilon(r) \ll 1$ (and will also be linear in M). You will then obtain equations of motion for $a(r)$, $b(r)$, and $\epsilon(r)$ by plugging into the Brans-Dicke field equations (for the metric and for the scalar field) retaining terms only to linear order in $a(r)$, $b(r)$, and $\epsilon(r)$. The Newtonian limit is taken by identifying $a(r) = 2GM/r$ (do you know/remember why? if not, think about it), and the PPN parameter γ is defined from $b(r) = 2\gamma GM/r$.

4. In a tour-de-force calculation, Hawking showed that, through quantum-mechanical processes, a Schwarzchild black hole emits a blackbody spectrum of radiation at a temperature $k_B T = \hbar/8\pi M$. This result can be understood in a fairly simple way. Recall that in the Feynman formulation of quantum mechanics, the amplitude for a system to take a path with an action S is $e^{iS/\hbar}$. Recall also that in statistical mechanics, the probability for a system at a temperature T to be found in a state with free energy S is $e^{-S/k_B T}$. This leads to the concept of finite-temperature field theory, which resembles quantum

field theory with an imaginary time that is periodic with period $t = (iT)^{-1}$. So all you have to do is the following. Start with the Schwarzchild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2.$$

Then make a transformation to a *Euclidean* Schwarzchild metric (with a positive signature) by making a change of variables to $\tau = it$. Your problem is then to find another change of variables that makes it clear that the imaginary-time coordinate τ in this Euclidean Schwarzchild metric is periodic with period $\tau = 8\pi M$.