General Relativity Ph236b
Problem Set 3
Due: In class, January 24, 2005

Suggested Reading: Ch. 8.7 in Carroll's book. Ch. 3 in Dodelson's book. Ch. 5.4 in Wald's book. Chs. 3-5 in Kolb and Turner's *The Early Universe*. Assorted sections in Peebles' *Principles of Physical Cosmology* cover what we're doing this week, but not in one place. Ch. 2 in Liddle & Lyth's *Cosmological Inflation and Large-Scale Structure*. There may also be relevant parts to Peacock's *Cosmological Physics*.

1. Suppose neutrinos have a mass of 1 eV. Then when their interactions freeze out in the early Universe at a temperature $T \simeq$MeV, they are relativistic, but today they are nonrelativistic. Find the energy distribution $f(K)$ (where K is the kinetic energy) of these cosmological neutrinos today.

2. As discussed in class, BBN can be used as a probe of possible deviations from the standard cosmological model and the standard model of particle interactions. The purpose of this problem is for you to work through some of these constraints. (a) Explain how the primordial ^{4}He abundance would change if the neutron lifetime were longer or shorter. (b) Suppose some particle theorists speculate that Fermi's constant G_F might actually be a function of time. What does BBN constrain the value of Fermi's constant to be at the time of nucleosynthesis? (c) It is plausible that there is a neutrino-antineutrino asymmetry in the Universe, and therefore that the cosmological neutrino mass density *today* is greater than it is in the canonical picture. What is the upper limit to the current neutrino density Ω_ν provided by BBN (assume the neutrinos are massless).

3. Consider a Universe that undergoes anisotropic expansion. Such a Universe has a metric

$$ds^2 = dt^2 - a_x^2(t)\, dx^2 \, + \, a_y^2(t)\, dy^2 \, + a_z^2(t)\, dz^2,$$

where $a_i(t)$ are scale factors of the three principal axes of the Universe. (a) Show that the Einstein equations for this metric lead to an analog of the Friedmann equation:

$$H^2 \equiv \frac{1}{9}\left(\frac{\dot{V}}{V}\right)^2 = \left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 = \frac{8\pi}{3m_{Pl}^2}(\rho + \rho_s),$$

where ρ is the ordinary energy density (radiation and nonrelativistic matter), and the shear "energy density" is defined to be

$$\rho_s \equiv \frac{m_{Pl}^2}{48\pi}[(H_x - H_y)^2 + (H_y - H_z)^2 + (H_x - H_z)^2].$$

Here $V = a_x a_y a_z$ is the "volume scale factor," $\bar{a} = V^{1/3}$ is the mean-scale factor, and the $H_i \equiv (\dot{a}_i/a_i)$ are the expansion rates of the three principal axes. (b) Next, show that the other Einstein equations (for $i \neq j$) become

$$\frac{d}{dt}\ln|H_i - H_j| = -3H = -3\frac{d}{dt}(\ln \bar{a}),$$

and that this implies that $\rho_s \propto \bar{a}^{-6}$. In other words, the effects of anisotropic expansion are mimicked by a new form of matter with energy density which decreases as $\bar{a}^{-6}$. (c) If ρ_s is too large, it will affect the results of BBN. Find the upper limit to ρ_s *today* provided by BBN. (d) Estimate how this BBN constraint to ρ_s might compare with the CMB constraint that $\Delta T/T < 10^{-5}$. Also, estimate how the BBN constraint might compare with ordinary measurements of the Hubble constant.

4. (a) Calculate the photon mean-free path and compare it with the Hubble distance H^{-1} at a redshift $z = 1500$, just before recombination. (b) After recombination, there will be a few residual electrons, due to freezeout of the recombination reaction $e^- + p \rightarrow H$. Calculate approximately the residual free-electron fraction. Make your assumptions about the capture cross section clear, and perhaps show how your result scales with that cross section. (b)In class we showed that a baryon-symmetric Universe would have a relic baryon density 10^9 times smaller than the observed density, from which we infer that there must be a cosmological baryon asymmetry characterized by a baryon-to-photon ratio today of $\eta = 5 \times 10^{-10}$. Calclate approximately the expected number of antibaryons in the entire observable Universe today, given this value for the baryon-to-photon asymmetry. Assume that the Universe remains perfectly homogeneous throughout its history. You may make reasonable approximations to get the answer correct to an order of magnitude or so.

5. In class I claimed that at a temperature $T \simeq 10^{19}$ GeV, classical general relativity will break down, and quantum-gravity will become important. Justify this with at least one simple (but correct) physical argument.