General Relativity Ph236b
Problem Set 4
Due: In class, January 31, 2005

Suggested Reading: Carroll, Ch 7.2; Parts of Chs. 4-5 in Dodelson; Ch. 14 in Liddle and Lyth's "Cosmological Inflation and Large-Scale Structure." Chapter 4 in Bertschinger's "Cosmological Dynamics"

1. Show that for the fluid stress-energy tensor,
$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} + \Sigma^{\mu\nu},$$
the stress tensor $\Sigma^{\mu\nu}$ can without loss of generality be taken to be traceless $\Sigma^\mu_\mu = 0$ and flow orthogonal $\Sigma^\mu_\nu u^\nu = 0$.

2. Consider the perturbed FRW spacetime,
$$ds^2 = a^2(\tau)\left\{-(1+2\psi)d\tau^2 + 2w_i d\tau dx^i + [(1-2\phi)\gamma_{ij} + 2h_{ij}]dx^i dx^j\right\},$$
and a particle with a coordinate three-velocity $v^i \equiv dx^i/d\tau$. Derive the four-velocity components (to linear order in perturbations and v) u^0, u_0, u^i, and u_i.

3. Show that Eqs. (4.32)-(4.34) in Bertschinger's article (astro-ph/9503125) are equivalent to Eq. (4.35)-(4.37). You can take $K = 0$ to simplify.

4. Evaluate the d'Alembertian in a flat, open, and closed FRW spacetime.

5. Verify Eq. (4.32) and Eq. (4.38) in Bertschinger's article.

6. Consider a flat FRW Universe with nothing but nonrelativistic matter with density ρ and pressure $p = 0$. Consider now small-amplitude perturbations to this FRW spacetime, and consider perturbations of small spatial size, $\lambda \ll H^{-1}$. Use some combination of the continuity and Euler equations (4.24-25 in Bertschinger) and the synchronous-gauge or Poisson-gauge (your choice) equations of motion (i.e., Einstein equations) to show that the amplitude of these small-amplitude small-wavelength perturbations grow in time as $\delta\rho/\bar{\rho} \propto a \propto t^{2/3}$ with the scale factor $a(t)$. You should only need the Einstein equations for the scalar potential h and ξ (in synchronous gauge) or phi and ψ (in Poisson gauge).